MAGNET HARDWARE

- $B_0$ field from superconducting magnet
- RF transmit/receive
- Gradient coils

$B_0 \rightarrow z$ (longitudinal)
$B_1 \rightarrow x, y$ (transverse)

$1 \text{T} = 10,000 \text{ Gauss}$
$\frac{1}{2\pi} = 42.6 \ \text{MHz/T}$

- (1) $B_0$ field
- (2) Body gradient coils
- (3) RF transmit body coil
- (4) RF receive-only head coils

Shim coil also embedded in here (not shown)

Circularly polarized $B_1$ field rotating $\frac{1}{2}$ to $B_0$ at Larmor freq. (B1 is several orders of magnitude smaller than $B_0$)

RF transmitter (10 kW)
RF receiver

Three one million watt amplifiers to add ramps to $B_0$ field

Usual water cooled
Spine & Precession

- nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers)
- moving charge creates magnetic field

Classical picture
- current loop from spinning charge (right-hand rule)
- N.B.: classically this would cause EM radiation, spindown

Stem-Gerlach experiment
- pass nuclei through strong mag. field \( \rightarrow \) split into just 2 beams

Microscopic picture
- no strong magnetic field \( B\phi = 0 \)
- all vectors same length, random directions
- Slight excess of "up" (3 ppm)
- \( x, y \) components still random
- Precissing vectors are "bunched" at any one moment around circle

Macroscopic picture
- bulk magnetization \( = 0 \)
- bulk magnetization \( = M_0 \)
- bulk magnetization precesses

Precession
- distinguish precession (slow) from spin (fast)
- treat classically, like spinning top

\[ 2\pi \omega = \frac{\gamma}{\hbar} B_0 \]

Larmor freq. (eg. 63 MHz)

\( \gamma \) = gyromagnetic ratio
\( h \) = Planck's constant

Bulk equilibrium magnetization (parallel to \( B_0 \))

\[ M_0 = |\vec{M}| = \gamma \frac{\hbar}{I(2I + 1)} B_0 N_s \]

where \( I = \frac{1}{2} \)

Two non-constants

\[ \frac{B_0}{N_s} = \text{i.e., } M_0 \text{ proportional to } B_0 \text{ strength} \]

\[ M_0 = \text{i.e., } M_0 \text{ proportional to number spins} \]

\( K = \text{Boltzmann const.} \)

\( T_s = \text{abs. temperature sample} \)
**Bloch Equation**

- **Time-dependent behavior of \( \vec{M} \) in the presence of an applied magnetic field (excitation \& relaxation)**
  - Change in magnetization vector
  - Precession: \( B = B_0 \)
  - Excitation: \( B = B_0 + B_1 \)

  \[
  \frac{d\vec{M}}{dt} = [\vec{M} \times \vec{B}] - \frac{\vec{M}}{T_1} - \frac{M_z}{T_2} \frac{\vec{M}_x \hat{i} + \vec{M}_y \hat{j} + \vec{M}_z \hat{k}}{T_2} - \frac{(M_z - M_z^0) \vec{M}}{T_1}
  \]

- **In the Larmor-rotating coordinate system, a tilt with a phase shift for a standard \( B_1 \) excitation is rotation around x-axis**

- **General case rotating**
  - (Distinguish \( B_1 \) \&, \( M \))

- **Longitudinal and Transverse relaxations**
  - \( \frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_z^0}{T_1} \)
  - \( \frac{dM_{xy}(t)}{dt} = -\frac{M_{xy}(t)}{T_2} \)

- **Solution to equations above**: Time-dependent free precession

  \[
  M_z(t) = M_z^0 (1 - e^{-t/T_1}) + M_z^0 (Q_+ e^{-t/T_1})
  \]

  \[
  M_{xy}(t) = M_{xy}(0) e^{-t/T_2}
  \]

  Given initial \( M_z^0 \), \( M_{xy}^0 \)

- **Lab frame: same!**

  - \( M_z(t) \) (equiv)
  - \( M_{xy}(t) \)

- **Lab frame: time e^{-t/T_2}**

- **Re-growing from \( 0 \)**

- **Leftover after pulse**

- **Decay**: \( M_z(t) = 63\% M_z^0 \)

- **Decay**: \( M_{xy}(t) = 37\% M_{xy}^0 \)

- **Decay**: \( \approx 100 \text{ min} \)

- **Decay**: \( \approx \text{1 sec} \)
**VECTOR ADD, MULTIPLY**

- adding vectors is easy
\[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
- just add components
- applies to complex numbers
- generalizes to any D
\[ \| \vec{c} \| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- multiple ways to multiply vectors: here are 3

**dot product**
(= inner product)
(= "scaled projection onto")
\[ c = \vec{a} \cdot \vec{b} = [b_x, b_y, b_z][a_x] = a_x b_x + a_y b_y + a_z b_z \]
(= scalar)
\[ c = \| \vec{a} \| \cos \Theta \]
\[ c = \| \vec{a} \| \| \vec{b} \| \cos \Theta \]
\[ c = \| \vec{a} \| \| \vec{b} \| \cos \Theta \rightarrow \text{zero if } \vec{a}, \vec{b} \text{ orthogonal} \]

**cross product**
(= outer product)
(can be generalized: see "geometric algebra")
\[ \vec{c} = \vec{a} \times \vec{b} = [0, b_x, b_y] [a_x] = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \]
(= vector)
\[ \vec{c} \rightarrow \text{orthogonal specific to 3D} \]
\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \sin \Theta \]
\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \sin \Theta \rightarrow \text{max if orthogonal} \]

**complex multiply**
(see also quaternions, geometric algebra generalization)
\[ \vec{c} = \vec{a} \cdot \vec{b} = [b_x, b_y] [a_x] = [a_x b_x + a_y b_y, a_x b_y - a_y b_x] \]
(= vector)
\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \]
\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \]

- angles add
- magnitudes multiply
- specific to 2D
\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \]
\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \]
\[ \sum \text{of angles: } \Theta_1 + \Theta_2 \]
**Simple Matrix Operations**

**Basic idea**
- A matrix \( \begin{bmatrix} \text{rotates} & \text{scales} \end{bmatrix} \) a vector

\[
\mathbf{b} = M \hat{a}
\]

**3D example**

\[
\begin{bmatrix}
\mathbf{b}_x \\
\mathbf{b}_y \\
\mathbf{b}_z
\end{bmatrix} =
\begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
\mathbf{d}_x \\
\mathbf{d}_y \\
\mathbf{d}_z
\end{bmatrix}
\]

**Add translate (after rotate/scale)**
- Commonly used "hack" for aligning solids
- A 4D matrix \( \begin{bmatrix} \text{rotates/scales} & \text{translates} \end{bmatrix} \) a 3D vector
- N.B.: Have to keep track of order!!
- Rotate/scale then translate ≠ translate, then rotate/scale
- Change rotation component: untranslate, rotate, retranslate

**3 special cases (3D):** rotate around each major axis without changing length (scale = 1.0)

- **Rotate around X-axis:**
  \[
  \mathbf{R}_x(\alpha) =
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \alpha & \sin \alpha \\
  0 & -\sin \alpha & \cos \alpha
  \end{bmatrix}
  \]
  e.g., 90° flip

- **Rotate around Y-axis:**
  \[
  \mathbf{R}_y(\alpha) =
  \begin{bmatrix}
  \cos \alpha & 0 & -\sin \alpha \\
  0 & 1 & 0 \\
  \sin \alpha & 0 & \cos \alpha
  \end{bmatrix}
  \]
  e.g., 180° flip to avoid adding 180° phase after 90° flip on x'

- **Rotate around Z-axis:**
  \[
  \mathbf{R}_z(\alpha) =
  \begin{bmatrix}
  \cos \alpha & \sin \alpha & 0 \\
  -\sin \alpha & \cos \alpha & 0 \\
  0 & 0 & 1
  \end{bmatrix}
  \]
  e.g., precession with B\(\phi\) along \(\mathbf{\hat{z}}\)

**General case**
- Rotate around general \(\mathbf{z}'\)-axis

\[
\mathbf{R}_{z'}(\alpha) = \mathbf{R}_z(\theta) \mathbf{R}_y(\phi) \mathbf{R}_z(\alpha) \mathbf{R}_y(\phi) \mathbf{R}_z(\theta)
\]

(Note: Quaternions are more efficient)
**Solutions to Simple Differential Eq.**

**Diff. Eq.:**
\[ dM_{xy}(t) = -\frac{M_{xy}(t)}{T_2} \]

**Solution:**
\[ M_{xy}(t) = M_{xy}(0) \cdot e^{-t/T_2} \]

**Goal:**
1) Find \( e_{xy} \) whose derivative satisfies diff. eq.
2) Also find sol'n (one of many) that passes thru init condition
   since our diff. eq. is:
   \[ \text{derivative of funct.} = \text{const. same funct.} \]
   \[ \rightarrow \text{try exponential, since derivative} (e^x) = e^x \]

**Deriv. Eq.:**
\[ \frac{dM_{xy}(t)}{dt} = -\frac{1}{T_2} \cdot M_{xy}(t) \]

**One sol'n:**
\[ M(t) = e^{-t/T_2} \]

**Take deriv. to check:**
\[ M'(t) = -\frac{1}{T_2} \cdot M(t) \]

**OK!**

**Another sol'n:**
\[ M(t) = \text{const.} \cdot e^{-t/T_2} \]

**Take deriv. to check:**
\[ M'(t) = -\frac{1}{T_2} \cdot \text{const.} \cdot e^{-t/T_2} \]

**OK!**

**Initial Condition:**
\[ M(t) = M_{xy}(0) \cdot e^{-t/T_2} \]

**Constant =**
\[ M_{xy}(0)\]
**Bloch Eq. - Matrix Version**

Differential Eq.:

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi
\]

Solution:

\[
\vec{M}(t) = \begin{bmatrix}
M_x(t) \\
M_y(t) \\
M_z(t)
\end{bmatrix} = \begin{bmatrix}
\cos wt & \sin wt & 0 \\
-\sin wt & \cos wt & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
M_x(0^+) \\
M_y(0^+) \\
M_z(0^+)
\end{bmatrix} = R_z(wt) \vec{M}(0^+)
\]

Include Relaxation

Differential Eq.:

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi - \frac{M_x \hat{e}_x + M_y \hat{e}_y}{T_2} - \frac{(M_z - M_z^0) \hat{e}_z}{T_1}
\]

Solution:

\[
\vec{M}(t) = \begin{bmatrix}
e^{i\frac{2\pi t}{T_2}} & 0 & 0 \\
0 & e^{i\frac{2\pi t}{T_2}} & 0 \\
0 & 0 & e^{-i\frac{2\pi t}{T_1}}
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\cos wt & \sin wt & 0 \\
-\sin wt & \cos wt & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
M_x(0^+) \\
M_y(0^+) \\
M_z(0^+)
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
M_z(0^+)(1 - e^{-i\frac{2\pi t}{T_1}})
\end{bmatrix}
\]
RF FIELD POLARIZATION

- Polarization (change of direction)
  - Linearly polarized field
    \[ \vec{B}_1 = B_1 \cdot \cos \omega t \hat{x} \]
  - Magnetic field (vs. electric field)
  - N.B.: \( \vec{B}_1 \) adds to much larger \( \vec{B}_0 \)
  - Circularly polarized field (quadrature)
    \[ \vec{B}_{1\text{circ}} = B_1 \left( \cos \omega t \hat{x} - \sin \omega t \hat{y} \right) \]
    \[ = B_1 \cdot e^{-i\omega t} \]
  - In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF spin

Typical 90° flip (around x = axis3)
(same coords as above, \( z \) at top)

Typical 180° flip (around opposite y-axis)
180° flip rezz
\( \sim 6 \times \) power at 90°
**SIGNAL EQUATION**

\[ \phi(t) = \int_{\text{obj}} \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

- Magnetic flux through coil (Scalar)

- For a particular instant in time vs. Block Mx\&B which is change & Mw\&t

- Local magnetization of object (Time-dependent)

- Position: \( \mathbf{r} = x, y, z \)

- Evaluate using free precession eqs. (Solution to Bloch) ignoring relaxation

- Rewrite with complex notation & time-dependence from lab frame Bloch

- Ignore the change in the z-component of the magnetization since it changes so slowly compared to the free precession of x- and y-components: \( \omega(\mathbf{r}) \gg 1/T_1(\mathbf{r}) \)

- This is why we can only record transverse magnetization, \( M_{xy} \), but not longitudinal magnetization (\( M_L \) changes too slowly, so \( V(t) = \frac{d\phi}{dt} \approx 0 \))

- Spatially-dependent resonant freq in rotating frame — i.e., after subtraction & \( \omega_0 = \gamma B_0 \)

**Laboratory frame Bloch solution:**
- \( M_L \) same
- \( M_T = M_{xy}(0) e^{i \omega_0 t} \)
- Demodulate

**Standard Signal Expression**

- Phase angle in rotating frame

- \[ \omega_0 = \text{radians/s} \]
- \( \omega_0 \) subtracted off by PSD

- \( \omega_0 \) is now scalar direction is here

- Sum across object

**S(t)**

\[ S(t) = \int_{\text{obj}} \mathbf{M}_{xy}(\mathbf{r}, 0) e^{-i \omega_0 t} \, d\mathbf{r} \]

- i.e., at a single point, RF signal is vector sum across object of local transverse magnetization vectors

- \( \omega_0 = \text{radians/sec} \)
- \( \omega_0 + \delta \omega \) are gradient
- \( \omega_0 + \delta \omega \) t = angle (\( \omega_0 + \delta \omega \) t = angle
- \( \omega_0 + \phi \) = angle

- Complex: 2 V(t)
**Phase-Sensitive Detection**

- How we get rotating frame:
  - Method for moving very high frequency Larmor oscillations down to tractable frequency range.

Demodulated signal \( \propto \) RF coil signal \& reference (transmitter):

\[
\propto \sin(w_0 + \delta w)t \cdot \sin w_0 t
\]

\[
\propto \frac{1}{2} \left[ \cos \delta wt - \cos (2w_0 + \delta w)t \right]
\]

This signal is digitized:
- Filter this one out w/ low pass filter.

**One freq - freq domain**

- Signal
- Reference
- Demodulated
- After filter

**Chirp - time domain**

- Chirp
- Center
- Demodulated
- No freq signal

- Two signals are made from a single receiving RF coil.
- A quadrature coil can be treated the same way (OK to combine after adding \( \frac{1}{2} \) phase, then PSD).
- Quadrature coil has better S/N since noise in each part is uncorrelated (\( \frac{1}{\sqrt{2}} \) better).

\[ S(t) \text{ complex} = e^{i \delta wt} \]

\[ S(t) \text{ real} \]

\[ S(t) \text{ imaginary} \]
**FID** - free induction decay, $T_2^*$

- **FID** (free-induction decay) from an RF pulse w/angle $\alpha$

$$S(t) = \sin \alpha \int_{w=-\infty}^{w=\infty} \rho(w) e^{-t/T_2(w)} e^{-i\omega t} dw$$

- For a single freq:

$$S(t) = M_2^0 \sin \alpha \ e^{-t/T_2} \ e^{-i\omega t}$$

- Max FID amplitude at $t=0$: $S(0) = M_2^0 \sin \alpha$

- If field inhomogeneous (Lorentzian distribution):

$$S(t) = \pi M_2^0 \gamma \Delta B_0 \sin \alpha \ e^{-t/T_2^*} \ e^{-i\omega t}$$

where

$$\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$$

| Basic FID envelope is proportional to $e^{-t/T_2^*}$

**NB:** actually complex; suggestive - actually a few million cycles here

- spin-spin & fluctuations fast not fixed w/time
- these $B_0$ drifts are static, so spin echo can cancel

- Continuous of $P$ frequencies (not Lorentzian)

- $T_2$ - unrecoverable, rapid (intrinsic)
- $T_2^*$ - recoverable, static added

- spectral density for "Lorentzian" mag. field inhomogen.

$$\rho(w) = \frac{M_2^0}{(\gamma \Delta B_0)^2 + (\omega - \omega_0)^2}$$

- For other dist, assume $T_2^*$ is exp. approx. e.g., see bottom

- Typical time course 10's of $\mu$s

- vs. each precession cycle, which is 10's of $\mu$s

- $\gamma \Delta B_0 \propto \omega_0$

- ignane space in object for now

- when all atoms interact same way in frequency ramps under curve
**ECHOES — spin echo**

- Just after 90° x' pulse, f_{lo} + f_{hi} have same phase.

- Relaxation + phase dispersion of f_{lo} + f_{hi} (both from B > B0).

- RF just tips vector(s) while retaining length. Relaxation includes tips and shrinks (and grows for echo).

- 180° x' pulse works, too, but echo will be +π phase (left side in figs above).

- Echoes generated even if second pulse not 180° (see next).

- FID decay (and echo growth/decay) described by T2*, from inhomogeneity.

- Reduction in height of echo compared to initial described by T2, echo fixes "star".
**ECHOES — Spin echo**

- \( \alpha_1 - \gamma - \alpha_2 \) (both pulses along \( y' \) for simplicity)

### Effect of \( \alpha \)-pulse

- \( M_x' \to M_x' \cos \alpha - M_z' \sin \alpha \)
- \( M_y' \to M_y' \)
- \( M_z' \to M_x' \sin \alpha - M_z' \cos \alpha \)

### Effect of \( \tau \) delay

- \( M_x' \to (M_x' \cos \omega \tau + M_y' \sin \omega \tau) e^{-\gamma/2} \)
- \( M_y' \to (-M_x' \sin \omega \tau + M_y' \cos \omega \tau) e^{-\gamma/2} \)
- \( M_z' \to M_z' (1 - e^{-\gamma T}) + M_z' e^{-\gamma T} \)

#### Immediately after \( \alpha_1 \) pulse

- \( M_x' (w, 0) = -M_z' (w) \sin \alpha_1 \)
- \( M_y' (w, 0) = 0 \)
- \( M_z' (w, 0) = M_z' (w) \cos \alpha_1 \)

#### After \( \tau \) delay

- \( M_x' (w, \tau) = -M_z' (w) \sin \alpha, \cos \omega \tau \ e^{-\gamma/2} \)
- \( M_y' (w, \tau) = M_z' (w) \sin \alpha, \sin \omega \tau \ e^{-\gamma/2} \)
- \( M_z' (w, \tau) = M_z' (w) [1 - (1 - \cos \alpha_1) e^{-\gamma T}] \)

#### Immediately after \( \alpha_2 \) pulse (no effect on \( M_y' \); rewrite \( x \) and \( y \) eq.s)

- \( M_{x'y'} (w, \tau) = M_z' (w) \sin \alpha_1 (\sin^{2} \alpha_1 \ e^{i \omega \tau} - \cos^{2} \alpha_1 \ e^{i \omega \tau}) e^{-\gamma/2} \)
- \( - M_z' (w) [1 - (1 - \cos \alpha_1) e^{-\gamma T}] \sin \alpha_2 \)

### Time dependent

- Free precession around \( z' \) (rewrite \( M_{x'y'} (w, \tau) \))

#### For a large num of freq's:

- \( M_{x'y'} (w, \tau) e^{-(t-\tau)/T_2} e^{-i \omega (t-\tau)} \)

- \( M_z' (w) \sin \alpha, \sin^{2} \alpha_1 \ e^{t/T_2} e^{-i \omega t} \)

- \( - M_z' (w) \sin \alpha, \cos^{2} \alpha_1 \ e^{t/T_2} e^{-i \omega t} \)

- \( - M_z' (w) [1 - (1 - \cos \alpha_1) e^{-\gamma T}] \sin \alpha_2 \ e^{-(t-\tau)/T_2} e^{-i \omega (t-\tau)} \)

#### Terms

- 1
- 2
- 3

#### Terms 1 & 3 are dephasing

- FID of echo

- Term 1 dephasing

- Nephase at \( t = 2T \)

### Echo signal

- Term 1

- Peak ampl.

### Echoes

- 90°-\( \gamma \)-90°
- \( S_1 (t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho (w) e^{-t/T_2} e^{-i \omega (t-\tau)} dw \)

### A_E

- \( S_2 (t) = \frac{\text{no 1/2 factor}}{\text{multiply by} \ i} \rightarrow \text{add} \ \gamma \text{phase} \)

- \( \rho (w) e^{-TE/T_2} dw \)

- \( S_2 (t) = \frac{M_z' \sin \alpha, \sin^{2} \alpha_1 \ e^{-TE/T_2}}{2} \)

### Echo amplitude, ignoring freq dependence of \( T_2 \)

- etc. in \( A_E \) ... like...
Echo TRAINS - spin-echo trains

- it's (too) easy to make echoes...

\[ E_n = \frac{3(n-1) - 1}{2} \]

- echoes after end of nth pulse
- 3 RF pulses \( \rightarrow 4 \) echoes (here)
- 6 RF pulses \( \rightarrow 12 \) echoes (!)

- a useful multi-echo sequence (CPMG) is a
  - 90° followed by 180° at 2\( \tau \) spacing

\[ \alpha_1 : M_L \rightarrow M_T \]
\[ \alpha_2 : \text{leftover } M_T \text{ flipped to } M_L \text{ (saved)} \]
\[ \alpha_3 : \text{flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays} \]
\[ \text{(after being held in limbo between } 180° \text{, FID}_2 \text{ and FID}_3) \]
\[ \text{acts like 2-pulse echo} \]

- typically, 90° and 180° applied in different axes (\( x', y', y'^* \))
  - which reduces phase errors due to imperfect 180° pulses
  - (since slightly-off rotation around \( y' \) affects phase less)
**EXTENDED PHASE GRAPHS**

- Using full Bloch Eq. solutions is tedious 😊
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize 90°, 180°)
- Problem #1: $\alpha$ pulse rotates a position of transverse magnetization into a position that results in rephasing $\Rightarrow$ regs QM view
- Problem #2: third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

Rule for effect of $\alpha$
**RF pulse on transverse mag**

Rule for effect of $\alpha$
**RF pulse on longitudinal mag**

Echo when phase path crosses zero
HYPER ECHOES

1. \(\alpha_2 - 180^\circ - \alpha_2 = \pm 180^\circ\)
   - 3 solid lines
   - 1 dashed line

2. \(\alpha_y = 180^\circ\)
   - Ignore amplitude
   - Surface of the sphere then defines a 2D space that you can move around in using:
     - Flip angle from \(z'(\alpha)\)
     - RF phase in \(x'y'(\phi)\)
     - "flip/phase" RF stim space

3. \(\alpha_0 - 180^\circ - \alpha_0 = 180^\circ\)

Practical use

- Multi-echo example
- Practical prob: 180° pulses deposit a lot of RF (6x 90°) 
  \(\rightarrow\) prob at high fields
- By arranging to get big echo in middle of \(k\)-space
  can get by with much less RF power

(N.B. mirror image coord syst vs. Bloch eq.)

Hennig & Scheffler (2001)
- Ignore amplitude
**Gradient Echoes** - $T_2^*$, GE chains

- **Initial negative gradient** dephases spins
- After $t = T$ of positive gradient, spins rephase
- Does not correct for $T_2^*$ inhomogeneities
  So echo amplitude is
  \[ A_E = e^{-t/T_2^*} \]
- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay

- Key difference between spin-echo (SE) and gradient echo (GE) is that $B_0$ inhomogeneities not canceled
  \[ \Rightarrow \text{hence, echoes are } T_2^*\text{-weighted, not } T_2\text{-weighted} \Rightarrow \text{more susceptible to inhomogeneities} \]

- Echo trains possible w/ gradient echo (CPMG - like)
- The faster the gradients are switched, the more echoes you get
- EPI hardware
  \[ \Rightarrow \text{64 echoes} \]
**IMAGE CONTRAST**

**T1 Saturation-recovery (no echo, just FID)**

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

RF

\[ \text{RF} \]

\[ 1 \quad \text{TR} \quad 2 \quad 3 \quad 4 \]

- Longitudinal magnetization
  \[ M_z \]

- Steady state after here

- Simple saturation/recovery w/no echo

- Initial conditions:
  \[ M_z \text{ before first pulse} = M_z^0 \]
  \[ M_z = 0 \text{ immediate after first pulse (i.e., 90° pulse)} \]

- From Bloch eq, \( M_z \) just before second pulse:

\[ M_z^{(2)}(O^{-}) = M_z^{(0)} (1 - e^{-TR/TI}) + M_z^{(0)} e^{-TR/TI} \]

\[ M_z \text{ before current pulse} \]

\[ M_z \text{ "regrowth-from-zero" term} \]

\[ M_z \text{ "left-immed.-after-pulse" term (i.e., decaying)} \]

- Given
  
  1. 90° pulse
  2. No \( M_{xy} \) left

  \[ \text{pure tip: } M_{xy} = M_z \]

- Tip existing mag

\[ M_z^{(m)}(O^{-}) = M_z^{(m)}(O_{+}) = M_z^0 (1 - e^{-TR/TI}) \]

\[ M_z \text{ longitudinal mag just before pulse} \]

\[ M_z \text{ transverse we can record after pulse} \]

\[ M_z \text{ transverse mag depends on TI}! \]

- That is, the not-completely-regrown longitudinal magnetization, which depends on TI, but which we cannot record, is completely converted to recordable transverse magnetization.

\[ I(r) = C \rho(r) (1 - e^{-TR/TI(r)}) \]

\[ \text{recon const, spectral dens} \]

\[ \rho(r) \text{ is p.density; underlies eqilib. } M_z^0 \]
**IMAGE CONTRAST**

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
  - because they are brighter than all the rest
  - because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
(e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state
  - for typical 1-2 sec
  - TR images reached
  - after 8 images
**IMAGE CONTRAST**  
IR (still just Saturation-recovery — no echo)

- **Inversion Recovery with no echo**

```
<table>
<thead>
<tr>
<th>50°</th>
<th>90°</th>
<th>TD</th>
</tr>
</thead>
</table>
```

**RF**

- "steady state" after here

**M₀** to **M₂**

- 180° pulse reverses longitudinal magnetization
  \[ M₂'(t) = -M₂^0 \]

- Recovery to end of first TI from long part of Bloch eq.
  \[ M₂' = M₂^0 \left( 1 - 2e^{-t/(2 T₁/T₂)} \right) \]

- Longitudinal then regrows from zero from second Bloch term only
  \[ M₂' = M₂^0 \left( 1 - e^{-(T₂-T₁)/T₁} \right) \]

- After second 180°, just change sign again
  \[ M₂' = -M₂^0 \left( 1 - e^{-(T₂-T₁)/T₁} \right) \]

- Apply relaxation eq. again
  \[ M₂' = M₂^0 \left( 1 - e^{-T₁/T₁} \right) - M₂^0 \left( 1 - e^{-\left(\frac{\text{TR}-T₂}{T₁}\right)} \right) \]

\[ M₂' = M₂^0 \left( 1 - 2e^{-T₁/T₁} + e^{-\frac{\text{TR}}{T₁}} \right) \]

\[ M₂' = M₂^0 \left( 1 - 2e^{-T₁/T₁} + e^{-\frac{\text{TR}}{T₁}} \right) \]

\[ M₂' \rightarrow \text{this is magnetization flipped to transverse, therefore made recordable} \]
- Steady state mag (2nd TR) just before 90°
  
  \[ M_z(t) = M_z^0 \left( 1 - 2e^{-\frac{(TR-TE/2)}{T1}} + e^{-\frac{TR}{T2}} \right) \]

- Echo signal (M_T) unlike in simple saturation-recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation

  \[ A_E = M_z^0 \left( 1 - 2e^{-\frac{(TR-TE/2)}{T1}} + e^{-\frac{TR}{T2}} \right) e^{-\frac{TE}{T2}} \]

- If we assume TE much less than TR, then we can simplify:

  \[ A_E = M_z^0 \left( 1 - e^{-\frac{TR}{T1}} \right) e^{-\frac{TE}{T2}} \]

  - Similar equation for SE-IR

  \[ A_E = M_z^0 \left( 1 - 2e^{-\frac{TI}{T1}} + e^{-\frac{TR}{T2}} \right) e^{-\frac{TE}{T2}} \]
\[ \alpha (\leq 90^\circ) \]

**GRE w/ small tip angle**

\[ \alpha \]

- Use basic longitudinal relaxation from Bloch eq. again

\[ \mathbf{M}_x^{(n-1)}(O_+) = 0 \quad \rightarrow \text{transverse dephased before next pulse} \]

\[ M_x^{(n)}(O_-) = M_x^0 (1 - e^{-TR/\tau_1}) + M_x^{(n-1)}(O_+) e^{-TR/\tau_1} \]

- Assume we have a small tip angle:

\[ M_x^0 \cos \alpha \]

\[ M_x^{(n)}(O_-) = M_x^0 (1 - e^{-TR/\tau_1}) + M_x^{(n-1)}(O_-) \cos \alpha e^{-TR/\tau_1} \]

- Assume we are in dynamic equilibrium:

\[ M_x^{(n)}(O_-) = M_x^{(n-1)}(O_-) = M_x^{ss}(O_-) \]

**Pre-pulse**

\[ M_x^{ss}(O_-) = \frac{M_x^0 (1 - e^{-TR/\tau_1})}{1 - \cos \alpha e^{-TR/\tau_1}} \]

**Post-pulse**

\[ M_x^{ss}(t) = \frac{M_x^0 (1 - e^{-TR/\tau_1}) \cdot \sin \alpha e^{-TE/\tau_2}}{1 - \cos \alpha e^{-TR/\tau_1}} \]

**Gradient echo amplitude**

\[ A_E = \frac{M_x^0 (1 - e^{-TR/\tau_1}) \sin \alpha e^{-TE/\tau_2}}{1 - \cos \alpha e^{-TR/\tau_1}} \]

TI contrast mostly depends on flip angle, not TR. \( \cos \alpha = 1 \) eliminates TI weight since \( \text{numerator} = \text{numerator} \).
**IMAGE CONTRAST**  
MDEFT / 3D FLASH

\[
\text{RF}_{\text{in}} \quad 90^\circ \quad \text{TD} \quad \alpha \quad \text{(spiral phase)} \quad \alpha' \quad 90^\circ \quad \text{TD} \quad k_1^\circ \quad \text{TI}
\]

- **Saturate**, wait for contrast, invert, wait for contrast, FLASH (continued)

A) \( M_2' \) (just after 90\(^\circ\)) = 0 (perfect 90\(^\circ\))

B) \( M_2' \) (after TD) = \( M_2^0 \left( 1 - e^{-\text{TD}/T_1} \right) \) (Bloch term \(^{91}\))

C) \( M_2' \) (just after invert) = \( \cos \phi \ M_2^0 \left( 1 - e^{-\text{TD}/T_1} \right) \)

D) \( M_2' \) (after TI) = \( M_2^0 \left( 1 - e^{-\text{TI}/T_1} \right) + \left[ \cos \phi \ M_2^0 \left( 1 - e^{-\text{TD}/T_1} \right) \right] e^{-\text{TI}/T_1} \)

\[
= M_2^0 \left[ 1 - \left( 1 - \cos \phi \left( 1 - e^{-\text{TD}/T_1} \right) \right) e^{-\text{TI}/T_1} \right]
\]

**Special Case TI=TD:**
\( M_2' = M_2^0 \left( 1 - e^{-\text{TI}/T_1} \right)^2 \)

- After the first RF pulse:

E) \( M_2' \) (just after final) = \( M_2^0 \left[ 1 - \left( 1 - \cos \phi \left( 1 - e^{-\text{TD}/T_1} \right) \right) e^{-\text{TI}/T_1} \right] \sin \alpha \)

\( \Rightarrow \) using hard 130\(^\circ\) inversion can cancel hard alpha B1 inhomogeneities (Thomas et al. '05)
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- **Signal-to-noise defined as:** \( \text{SNR} = \frac{\text{avg signal}}{\text{s.d. noise}} \)
- **Temporal SNR:** \( \frac{\text{SNR}}{\sqrt{\text{temporal SNR}}} \)
- "Contrast" is a difference
- **Contrast-to-noise ratio:**
  \[
  \text{CNR}_{AB} = \frac{\overline{S_A} - \overline{S_B}}{\sigma_n} = \text{SNR}_A - \text{SNR}_B
  \]
  e.g., WM - GM (activated vs. rest)

- **Spin-echo:**
  \[
  A_E = M_o \left(1 - e^{-TR/T1}\right) e^{-TE/T2}
  \]

---

**Gradient echo:**

\[
A_E = \frac{M_o \left(1 - e^{-TR/T1}\right) \sin \alpha}{1 - \cos \alpha e^{-TR/T1}} e^{-TE/T2*}
\]

- Long TR
- Short TR

**General rules:** Spin-echo, long TR GE

<table>
<thead>
<tr>
<th>Proton-density weighted</th>
<th>TR HH (no T1 diffs)</th>
<th>TE HH (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>TR HH (big T1 diffs)</td>
<td>TE HH (no T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>TR HH (no T1 diffs)</td>
<td>TE HH (big T2 diffs)</td>
</tr>
</tbody>
</table>
**SIGNAL-TO-NOISE S/N**

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR per voxel } \propto \Delta x \Delta y \Delta z \sqrt[3]{\text{Na}_z \text{N}_x \text{N}_y \text{N}_z \Delta t} \\
\text{voxel size} \quad \text{num repeats} \quad \text{special number of read} \quad \text{read timestep}
\]

- Size (volume) of voxels (with the number of voxels held constant), linear effect on S/N

  \[\text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better } S/N\]

- More voxels (with size of voxels, \(\Delta t\) per read step constant), \(\sqrt{n}\) effect on S/N

  \[\text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{\sqrt{128 \times 128}}{\sqrt{64 \times 64}} = 2 \text{ times better } S/N\]

- # acquisitions, \(\sqrt{n}\) better S/N

  \[\text{e.g., } 1 \text{ acq } \rightarrow 2 \text{ acq } \rightarrow \frac{\sqrt{2}}{1} = 1.41 \text{ times better } S/N\]

- Larger timestep during readout, \(\sqrt{\Delta t}\) better S/N

\[\Delta t = \frac{1}{\text{BW}_{\text{read}}}, \text{ digitization timestep during echo acquisition}\]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog low-pass filter
- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \(\Delta t\)
- Must filter out freq's \(> f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
**COMPLEX ALGEBRA**

Real/Imaginary

- add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
- mult: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

Angle/Phase

- add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
- mult: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2, \phi_1 + \phi_2)\)
- complex to real power: \((A, \phi)^n = (A^n, n \phi)\)

Real to Complex Power

\[ e^{i \phi} = \begin{cases} \text{expands as series} \\ \text{recognize \( \cos, \sin \) series} \end{cases} \]

- \[ e^{i \phi} = \cos \phi + i \sin \phi \]
- \[ e^{i \phi} = \cos \phi, \sin \phi \]
- \[ e^{i \phi} = \text{vector on unit circle} \]

\[ e^{i \phi} = (\cos \phi + i \sin \phi)^n = \cos n \phi + i \sin n \phi \]

**Fourier Transform**

- \[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-j 2\pi ft} dt \]
- \[ H(\omega) = \int_{-\infty}^{\infty} h(t) e^{j 2\pi \omega t} dt \]

**Convolution Theorem**

\[ \mathcal{F}[g(x) \ast h(x)] = \mathcal{F}[g(x)] \cdot \mathcal{F}[h(x)] \]

**Convolution**

- \[ f(x) = g(x) \ast h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x-z) dz \]
- \[ f(x) = g(x) \otimes h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x+z) dz \]

- **The Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transform of each function.**

- **Convolutions in the frequency domain are multiplications in the time domain.**

- **The convolution theorem** states that the Fourier transform of the convolution of two functions is the product of their Fourier transforms.

- **Convolution** is a fundamental operation in signal processing and is used in various applications, such as image processing and audio processing.

- **Fourier transform** is a mathematical tool used to analyze signals and functions by decomposing them into a sum of sinusoidal waves.

- **Frequency domain** refers to the representation of signals or functions in terms of frequency components.

- **Time domain** refers to the representation of signals or functions in terms of their time variations.

- **Phase shift** is a phase difference between two signals.

- **Impulse response** is the output of a system when the input is an impulse.

- **Spectral analysis** is the process of analyzing signals in the frequency domain.

- **Frequency spectrum** is the distribution of a signal's power across different frequencies.

- **Complex valued function** is a function that maps points in a complex plane to complex numbers.

- **Complex exponent** is an exponential function of a complex variable.

- **Complex conjugate** is the complex number with the same real part and an imaginary part equal in magnitude but opposite in sign.

- **Complex number** is a number that can be expressed in the form a + bi, where a and b are real numbers and i is the imaginary unit.

- **Complex signal** is a signal that can be represented by complex numbers.

- **Complex variable** is a variable that can take on complex values.

- **Complex waveform** is a waveform that can be described by complex numbers.

- **Complex wave** is a wave that can be expressed in terms of complex numbers.

- **Complex vector** is a vector that can be represented by complex numbers.

- **Complex number system** is a system that deals with complex numbers.

- **Complex plane** is a plane where each point represents a complex number.

- **Complex analysis** is the study of functions of complex variables.

- **Complex integration** is the process of integrating complex-valued functions.

- **Complex differentiation** is the process of differentiating complex-valued functions.

- **Complex Taylor series** is a series representation of a complex function.

- **Complex Fourier series** is a series representation of a periodic complex signal.

- **Complex Laplace transform** is a transform used to analyze linear time-invariant systems.

- **Complex Laplace domain** is the representation of signals or functions in the complex frequency domain.

- **Complex convolution** is a convolution in the complex domain.

- **Complex correlation** is a measure of similarity between two complex-valued signals.

- **Complex cross-correlation** is a measure of how similar two complex-valued signals are.

- **Complex autocorrelation** is a measure of how similar a complex signal is to itself.

- **Complex impulse response** is the response of a system to an impulse.

- **Complex transfer function** is the ratio of the output of a system to the input.

- **Complex system theory** is the theory of systems that deal with complex variables.

- **Complex system analysis** is the process of analyzing complex systems.

- **Complex system design** is the process of designing complex systems.

- **Complex system control** is the process of controlling complex systems.

- **Complex system optimization** is the process of optimizing complex systems.

- **Complex system simulation** is the process of simulating complex systems.

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Fourier transform (1)

\[ H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j2\pi ft} \, dt \]

- How to calculate \( H(f) \) for one \( f \) (\( f=3 \)):

Real signal

- Real frequency domain

Imaginary signal

- Imaginary frequency domain

\[
\begin{bmatrix}
\cos \\
c^{-j3t} \\
\sin \\
\end{bmatrix}
\]

- Cosine and sine functions

\( e^{-j3t} \)

- Complex multiply

Integrate/sum these multiplies across all \( t \)

- Real frequency domain

Imaginary frequency domain

- Complex frequency domain

Cartesian (\( r, \theta \))

- Polar coordinates

Frequency domain

- Amplitude

Phase

- Frequency domain

Like correlating with \( \sin \) and \( \cos \) (at each freq) so we get phase (at each freq).
Fourier transform (1b)

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} = \cos (-\phi) + i \sin (-\phi) = \cos \phi - i \sin \phi \]

- \( \cos \phi \) is an "even" function, \( \sin \phi \) is an "odd" function.

**even**

\[ \cos(x) \]
\[ \downarrow \text{equals} \]
\[ \cos(-x) \]
\[ \downarrow \text{flips} \]
\[ -\cos(x) \]

**odd**

\[ \sin(x) \]
\[ \downarrow \text{flips} \]
\[ \sin(-x) \]
\[ \downarrow \text{equals} \]
\[ -\sin(x) \]

If a function is mirror-symmetric along the x-axis around zero:
- \( f(x) = f(-x) \)
- It is even

**An orthogonal decomposition**

- Think of discretely sampled \( \sin(bx) \), \( \cos(bx) \) as vectors.
- \( \text{Corr}(\vec{v}_1, \vec{v}_2) \equiv \text{projection of } \vec{v}_1 \text{ onto } \vec{v}_2 \equiv \vec{v}_1 \cdot \vec{v}_2 \)

\[
\begin{align*}
\text{Corr}(\cos bx, \sin bx) &= 0 \\
&= \text{Sin & cos of same frequency are orthogonal} \\
&= \sin 2x \cos 2x
\\
\text{Corr}(\sin bx, \sin bx) &= 0 \\
&= \text{different integer frqys of sin (or cos) are orthogonal} \\
&= \sin 2x \sin 3x
\\
\text{Corr}(\cos bx, \sin bx) &= 0 \\
&\text{[as above]}
\end{align*}
\]

In the continuous case, orthogonal functions defined as:
\[
\int_{-h}^{h} g(x) dx = 0
\]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

- Start with spike in image domain
- Take example of spike at $x = 0$
  \[
  \begin{bmatrix}
  \cos(x) \\
  \cos(2x) \\
  \cos(kx)
  \end{bmatrix}
  \text{all freqs correlate w/ spike at } x = 0
  \]

- If spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates

- To see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the $e^{-j2\pi kx}$ cos and sin at location of spike

- $\cos$ vs. $\sin$:
  - $\cos$: (even)
  - $\sin$: (odd)

- Positive spikes same dist from origin: $\Rightarrow$ pick cos's
- Positive & negative spikes, same dist: $\Rightarrow$ pick sin's

- This is one way of thinking about what one point in K-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

1. Fourier Transform
   - Real image
   - Imaginary image
   - Real spatial freq.
   - Imaginary spatial freq.

2. Amplitude image
   - Phase image
   - Amplitude spatial freq.
   - Phase spatial freq.
   - What you see on screen: view complex vectors directly

3. Complex vectors
   - Zero vectors
   - Complex vectors

- 3 equivalent representations of image & spatial freq. space
Fourier Transform of an Image (3)

- What a single k-space point looks like in image space (polar coordinates $A, \phi$ instead of $r, i$)

- Cartesian dimension of k-space — x- and y- spatial freq

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin, cos — don’t confuse $k_x, k_y$ w/ sin, cos!

**Inverse Fourier Transform**

(should be all zero not same as "stripe phase" above)

(N.B. — increasing one 1D component increases the spatial freq of the 2D wave and rotates it)
FOURIER Transform of Image (4)

- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)
- Example: cosinusoid in image space, then shifted in x-dir

Real Image

\[ I(x, y) = \cos(x) \]

FT of Real Image

\[ I(x, y) = \cos(x - \frac{\pi}{4}) \]

\[ \rightarrow \text{halfway between} \cos \text{ and } \sin \text{ (shifted 45° to right)} \]
FOURIER TRANSFORM OF IMAGE (S)

- (cont.) center of k-space (real image)
- complex image

**REAL IMAGE**

\[ I(x,y) = 1 + \cos(x) \]

![Graph of real image](image)

Center of k-space:

\[ H(k) = \int x h(x) e^{-i2\pi kx} \]

Avg image brightness \( \leq 1 \) (real)

\[ \text{FT of real image} \]

Positive center k-space

\[ \text{FT} \]

\[ \text{FT}^{-1} \]

\[ \text{complex} \]

\[ \text{Max} \]

\[ \text{complex} \]

- the center of k-space is zero w/ pure sin or cos image b/c avg brightness = 0

**COMPLEX IMAGE**

\[ I(x,y) = \cos(x) - i \sin(x) \]

\[ e^{-i2\pi kx} \]

\[ \text{FT of complex image} \]

\[ \text{ft, ft}^{-1} \]

\[ \text{complex conjugate} \]

\[ \text{non-Hermitian: k-space will not have Hermitian symmetry if image is real} \]

\[ \text{complex conjugate} \]

\[ \text{complex number w/ sign flipped in imag. part} \]

\[ \text{is equal to new value w/ new arg:} \]

\[ H(-x) = H^*(x) \]

\[ \text{N.B., this is like what a gradient does!} \]
GRADIENT COILS

- gradient coils for x, y, z generate approximately a linear gradient in the strength of the $z$-component of the magnetic field $B_z$

- for example, the $x$ gradient coil induces a ramp in $z$-component of the magnetic field when moving in the $x$-direction:

$$B_{G,z} = G_x x$$

* - since a pure linear gradient of $B_{G,z}$ in only the $x, y,$ or $z$ direction is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the $x$- and $y$-direction ($B_{G,x}$ and $B_{G,y}$)

- the other magnetic field components are usually ignored because they are so small relative to $B_{G,z}$, since $B_{G,z}$ is added to $B_0$, and since $B_0$ is much stronger than $B_{G,z}, B_{G,y},$ and $B_{G,x}$

- since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced

- the Maxwellian terms $B_{0,x}, B_{0,y}$ are known; can be included in the reconstruction process

$$\Delta \phi_G(x) \approx -\frac{x^2 G_x^2 t}{2B_0}$$
SLICE SELECTION ($G_z$)

- Slice select gradient on during RF stim

$$B_z \rightarrow \frac{\gamma}{2\pi} G_z Z$$

- Protons here can only be excited by a narrow band of radio frequencies

- To apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

  $$\text{in practice, Gaussian pulse envelope good too}$$

  $$= \frac{\sin(\theta)}{\theta}$$

  $$\xrightarrow{\text{Fourier Transform}}$$

  $$\text{this excites protons in a narrow slab}$$

- Since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode), these have to be removed by a post-excitation rephasing $Z$-gradient

  $$\text{RF} \quad \text{90°}$$

  $$\text{approximation from assuming tip occurs instantaneously in middle}$$

  $$\text{valid for small tip: 90° \rightarrow 52%}$$

  $$\text{in practice: adjust to max, use crusher to kill spurious transverse on 180°}$$
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx. even tho tipping is non-linear)

\[ B_1(t) \propto \int_{-\infty}^{\infty} p(f) e^{-i2\pi ft} df \]

- Time dependent RF stimulation (complex)
- Solve with: \( p(f) = \text{frequency band} \)

\[ B_1(t) = A \cdot f_w \cdot \text{sinc}(\pi f_w t) e^{-i2\pi f_c t} \]

- Amplitude controlling flip angle
- Freq. width controlled by slice width (N.B. wider \( f_w \) is narrower sinc)
- Sinc envelope determined by freq. width \( f_w \)
- Modulation (complex) at center freq. \( f_c \)

Fourier Transform Pair, Rules:
- Convolution in one domain is multiplication in the other
- Convolution with delta funct. impulse moves function to impulse center

Fourier Transform Solution to: \( \frac{\pi}{\theta} \)
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) chemical shift change freq  →  gradient changes freq.</td>
<td></td>
</tr>
<tr>
<td>2) stimulate w/ broadband RF  →  same</td>
<td></td>
</tr>
<tr>
<td>3) time-sample FID containing multiple freqs  →  same</td>
<td></td>
</tr>
<tr>
<td>4) FT of FID to get spectrum  →  FT of FID to get Δx offsets</td>
<td></td>
</tr>
</tbody>
</table>

- this is technically correct (FT of FID) but highly misleading

  e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican Turn"

  idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift like frequencies

  rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations, (which are analogous to multiple time points)

- i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>spectroscopy</th>
<th>signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>samples of oscillations in time-domain</td>
<td>FT → frequency-domain spectrum of shifts</td>
</tr>
</tbody>
</table>

MRI

<table>
<thead>
<tr>
<th>signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>samples of spatial freq. in freq. domain</td>
<td>FT⁻¹  →  spatial object (like a time-domain signal)</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because FT⁻¹ FT = FT⁻¹ (except sign change)
FREQUENCY ENCODING (1)  

- Frequency encode gradient ($G_x$) causes precession rates to vary linearly in x-direction

\[ \text{precession rate of } B_z \text{ in } x \text{-direction} \]

\[ x \rightarrow \]

\[ \leftrightarrow \text{ correct} \] (remember that strength of $G_x$ causes variation of slope of $B_z$ in x-direction)

- Different frequency signals are mixed together and recorded as a 1-D signal over time

\[ \leftrightarrow \text{ correct, but remember, we are recording summed local magnetization vectors after de-modulation} \]

- A Fourier transform, which converts back and forth between x-position (cf. time) and spatial frequency (cf. temporal freq) is done on signal

\[ \leftrightarrow \text{ correct} \]

- Spatial frequencies get confused/confused with precession frequencies

\[ \leftrightarrow \text{ wrong} !! \]

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

\[ \leftrightarrow \text{ conceptually wrong} !! \]

\[ \leftrightarrow \text{ actually converts spatial frequencies to spatial position} \]

\[ \rightarrow \text{ the spatial frequency increases for each time point in the readout} \]

\[ \rightarrow \text{ the precession freq ramp is constant each timestep} \]

\[ \star \]

N.B.: gradient ramp does not need to be exactly the same as recording
FREQUENCY ENCODING (2)

- "frequency"-encode gradient \((G_x)\) turned on during
  
during echo causes precession rates to immediately vary with \(x\)-position

- at beginning of gradient on, the phase of signal coming from each \(x\)-position is the same
  
  **Summed phase angle is what we measure**

- early after gradient on, phase advances (because of faster precession frequency) arise with greatest
  
  phase advance at largest \(x\)-position

- later during gradient on, phase advances cause multiple wraparounds of phase angle across space

- in practice, the lowest spatial frequency \((= 0)\) occurs in the middle of the gradient on time
  
  because the phase is "rounded" negatively by a preparatory gradient (to the highest negative spatial frequency) before data collection occurs

\[ b = \text{max positive} \]

\[ b = \text{max negative} \]
FREQUENCY ENCODING (3) why each datapoint is 1 spatial freq

Standard Fourier transform: (Temporal freq ↔ time)

\[ H(\omega) = \int_{t=-\infty}^{t=\infty} h(t) \cdot e^{-i 2\pi \omega t} dt \]

"k" is often used instead of "f" for the frequency variable

Imaging equation: (spatial freq. ↔ space)

\[ S(\omega) = \int_{x=-\infty}^{x=\infty} I(x) \cdot e^{-i 2\pi \omega x} dx \]

Sum across x of object

- Oscillations come from readout phase wrapping, where \( f \) is single spatial freq (e.g., 5) and \( x \) goes across object
- This is done by RF coil recording sum

Get this single readout point by summing signal across x-position (RF coil records sum)

even though variable is \( \omega \), it represents one time point during readout

to make image, do inverse Fourier transform of recorded signal \( S(\omega) \)

END: \( G_x(t-TE) \)

SE: \( G_x(TE) \)

\( \omega = G_x \), that is, spatial freq depend on amount of time gradient was on (this \( \omega \) increases with time!)

don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each \( x \) position)
ALTERNATE DERIVATION (incl. effects of $G_x$) SIGNAL EQ

- Oscillators at $w = \gamma B$ at each position (just $x$ for now)

$$S(t) = m(x) e^{-i\phi(x)} dx$$

- By definition, freq. $w$ is rate of change of phase, $\phi$

$$\frac{d\phi(x,t)}{dt} = w(x,t) = \gamma B(x,t)$$ and integrating

$$\phi(x,t) = \int_0^t w(x,t) dt = \int_0^t B(x,t) dt$$

- Assuming phase initially 0, B affected by gradients

$$B(x,t) = B_0 + G_x(t) x$$

$$\phi(x,t) = \gamma \int_0^t B_0 dt + \left[ \gamma \int_0^t G_x(t) dt \right] x$$

$$= \omega_0 t + 2\pi k_x(t) x$$

$k$ is time integral of gradient waveform

- Demodulation removes the $B_0$-caused carrier frequency $e^{-i\omega_0 t}$ from the first equation

$$S(t) = \int_x m(x) e^{-i \omega_0 t} \frac{2\pi k_x(t) x}{dx}$$

Amplitude of each oscillator

Gradient-controlled phase
- Turn on gradient after excitation but before readout.
- Different levels of $G_y$
  \[ B_{2y} \quad \text{y} \quad \text{y} \quad \text{y} \]
- Higher levels of $G_y$ (slope of $B_z$ in y-direction!)
  \[ \Rightarrow \text{higher spatial freq. (more phase wraps) in y-direction} \]
- Phase wraps persist after phase-encode gradient off
- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase

**2D Imaging Equation**

\[
S(k_x, k_y) = \int \int I(x, y) \cdot e^{-i2\pi(k_x x + k_y y)} \, dx \, dy
\]

- Signal recorded at single time point (one readout point)
  \[ \downarrow \text{complex signal (from phase-sensitive detection)} \]
  \[ \downarrow \text{done by RF coil} \]
  \[ \text{sum across x-y plane} \]
  \[ \text{image (strength of magnetization at each x-y pt)} \]
- Phase (vector of unit length and particular angle which is function of $G_x$ and $G_y$)
  \[ \downarrow \text{scalar (what we try to reconstruct)} \]
  \[ \text{phase angle (of scalar magnetization! in rotating frame, set by gradients) increases each TR} \]

- Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia"—they stay wherever the gradients last left them.
3-D IMAGING — two phase-encode gradients

- Use $z$-gradient for 2nd phase-encoding instead of slice selection

- Excitation of whole slab (slice-select is whole brain)

- Simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [3PGR]

\[
S(k_x, k_y, k_z) = \int \int \int \text{signal recorded at single point of readout}
\]

\[
I(x,y,z) = e^{-i2\pi(k_x x + k_y y + k_z z)} \ dx \ dy \ dz
\]

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice

phase stripes created throughout volume vs. slice:

N.B., this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

Since the phase-encode gradient and the freq-encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

- Successive read-out steps:
  - Small phase encode $G_y$
  - TE
  - Large phase encode $G_y$
  - TE

More rotation with higher spatial freq.

- 3D phase encode with $G_y$ and $G_z$, starts rotated in y-z plane.

N.B.: Stripes have sharp edges from phase wrap (not sinusoid since q from 2-comp. quadrature!)

Stripes here represent complex value

Phase of whole image summed to one (complex) number by RF coils

e.g., after x-gradient, spins at a point might be 3 cps ahead while after y-gradient spins at same point 7 cps ahead; but counting wraps in x-direction, still only 3 ahead.
GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point.

\[ k = \int_0^{t_{rec}} G(t) \, dt \]

Spatial freq recorded as gradient strength as function of \( t \)

- all of the following gradients end up at the same point in k-space:

Frequency-encode (FID)

RF

\[ 90^\circ \]

\[ G_x \]

\[ \text{samples} \]

\[ \rightarrow \]

\[ k_y \]

\[ k_x \]

Frequency-encode gradient echo

RF

\[ 90^\circ \]

\[ G_x \]

\[ \rightarrow \]

\[ k_y \]

\[ k_x \]

\[ \text{neg} \, G_x \]

Frequency-encode spin-echo (plus gradient echo!!)

RF

\[ 90^\circ \]

\[ 180^\circ \]

\[ \tau \]

\[ G_x \]

\[ \rightarrow \]

\[ k_y \]

\[ k_x \]

\[ \text{pos} \, G_x \]

(Next 180° moves to conjugate point)

Phase-encode then frequency encode gradient echo

RF

\[ 90^\circ \]

\[ G_y \]

\[ G_x \]

\[ \rightarrow \]

\[ k_y \]

\[ k_x \]

\[ G_x + G_y \]
**IMAGE RECONSTRUCTION**

\[
S(k_x, k_y) = \iint I(x, y) e^{i2\pi (k_x x + k_y y)} dx \, dy
\]

\[
I(x, y) = \iint S(k_x, k_y) e^{i2\pi (x k_x + y k_y)} dk_x \, dk_y
\]

- Ideally \( I \) is real, but in practice it is complex. To use the amplitude image, take the absolute value:

\[
|A| \rightarrow 0
\]

Adding exponents is the same as multiplying two \( e^{i2\pi k_x x} \) terms.

Same as two sequential 1D FFTs (actual code):

\[
= \int_{k_y} S(k_x, k_y) e^{i2\pi k_x x} e^{i2\pi k_y y} dk_x \, dk_y
\]

In practice, a finite number of samples, \( N \) and \( M \), are collected, and \( k_x \) and \( k_y \) directions of \( k \)-space (integral \( \rightarrow \) discrete sum)

\[
I(x, y) = \sum_{n = -N/2}^{N/2-1} S(n, m) e^{-i2\pi n \Delta k_x x \Delta k_x} e^{i2\pi m \Delta k_y y \Delta k_y}
\]

Sampling interval in \( k \)-space.
**Sampling**

- Must consider effects of limited points in k-space.
  - Limited in range of frequencies sampled (k_min -> k_max)
  - Limited in rate of sampling (ΔK)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling.

- Correct reconstruction
  - Infinite frequency range
  - Infinitely fine sampling

- Correct plus replicas
  - Infinite frequency range
  - Finite spacing of samples

[as above w/ blurring, ringing]

- Aliasing occurs in spatial domain
  - Replicas overlap, causing wraparound

Thus, finer sampling of same range of spatial freqs increases FOV.
**UNDER/OVER SAMPLE**

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]
\[ \delta_x = \text{FOV}_x = \frac{1}{N \Delta k_x} \]

- *FOV* (distance to repeat) is reciprocal of spatial frequency sampling interval
- Pixel size is FOV divided by *K-space* sample count

### More Examples (not incl. less samples to same spat. freq [bottom last page])

1. **Basic Image**
2. **Same num samp. to 2x spat. freq.** (i.e., gradients stronger or time ON longer)
3. **2x num. samples to same spat. freq.** (i.e., gradients weaker or time ON shorter)
4. **2x number samples to 2x spat. freq.** (i.e., gradients stronger or time ON longer)

---

- Basic image
- Square pix
- X-pix half width
- Replicas intrude
- Scanner makes square image wrap occurs
- Square pix
- Twice x-pix count so FOV = 2x
- This is "phase oversamp"
- Scanner crops to square replicas move out
- X-pix half width
- Twice x-pix count
- Same FOV
- This is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. Sample data in spatial frequency space

\[ \text{\ldots} \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \text{\ldots} \]

\[ \times \text{ multiply} \]

\[ \text{\ldots} \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \text{\ldots} \]

\[ = \text{equals} \]

\[ \text{\ldots} \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \text{\ldots} \]

Useful FTs

**Rect**

\[ \text{Rect} \left( \frac{x}{w} \right) \rightarrow W \text{sinc}(\pi W k) \]

**Gaussian**

\[ e^{-\pi x^2} \rightarrow e^{-\pi k^2} \]

**Comb**

\[ \sum_{n=-\infty}^{\infty} \delta(x - n/\Delta k) \rightarrow \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) \]

Limit approach to Fourier transform of comb

\[ \text{FOV} = \frac{1}{\Delta k} \]

\[ \Delta k = \frac{1}{\text{FOV}} \]
**Point-Spread Function**

\[
\hat{I}(x) = \Delta k \sum_{n} S(n\Delta k) e^{i 2\pi n \Delta k x}
\]

- Set true image to \( S \)-function, then measured signal is:
  \( S(n\Delta k) = 1 \)

- Substitute into \( \hat{I} \) to get PSF:
  \[
h(x) = \Delta k \sum_{n} e^{i 2\pi n \Delta k x}
  \]

- Simplify
  \[
h(x) = \Delta k \frac{\sin(\pi N \Delta k x)}{\sin(\pi \Delta k x)} \Rightarrow \text{periodic!}
  \]

- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in \( k \)-space is an image sinc.

![Image](image.png)

How PSF modifies ideal (infinite \( k \)) image

- Convolve
- \( \ast \) ringing
- Fourier
- multiply
- acquisition window (truncated high spatial)

Central replica same as sinc
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \int_x I(x) e^{-i 2\pi k_x x} dx \]  
Signal eq. \( \rightarrow \) fwd problem

\[ I(x) = \int_k S(k_x) e^{+i 2\pi k_x x} dk_x \]  
Recon eq. \( \rightarrow \) inv. problem

\[ s = F i \]

Linear "forward solution"  
Matrix vectors have complex entries  
Can build in any measurable priors

\[ F_{x,y,z} = g(x,y) e^{-\frac{(nT \pm m\Delta T + TE)}{T_2}} e^{-i \gamma B_0 (nT \pm m\Delta T)} e^{-i \gamma B_0 (n\Delta k_x x + m\Delta k_y y)} \]

- T2 decay  
- B0 error  
- T2 + phase

Multi-coil

\[ s \]

\[ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} F_{coil 1} \\ F_{coil 2} \end{bmatrix} i \]

- Naturally incorporates undistorted field map  
- Different sensitivity function for each coil  
- Contains additional info about source loc.  
- But, need reference scan, lo-res ok  
- (need phase corrections for each coil ?)

\[ i = F^+ s \] over-determined

More Fourier inverse

\[ F^+ = (F F^T)^{-1} F^T \]

(x,y)^2 \( \rightarrow \) "small"  
(x,y,coil)^2 \( \rightarrow \) 16x bigger  
for 4 coils

\[ i = [(F F^T)^{-1} F^T] s \]  
Slice-by-slice  
Assume slice select swamps ABG
FAST SPIN ECHO (FSE)  RARE, FSE, 3DFSE

- one 90° pulse followed by multiple 180° pulses (e.g., 8)
  each with a different phase-encode gradient
- each phase "winder" is "unwound" because leftover phase would
  be re-focused away by 180° (vs. EPI where it persists between blips)

2DFSE

RF

G_x 

G_y

G_z

Signal

- the "effective TE" is the TE when center of k-space
  is collected (largest effect on contrast, largest echo)
- each subsequent echo has more T2 decay: \( E_n = e^{-n\text{TE}/T2} \)
  \( n = 1, 2, \ldots M \)
- by arranging to collect \( k_y = 0 \) early, PD-weighted instead of T2-weighted

- possible to correct different T2-weighting of echoes by
  estimating T2 curve from \( G_y = 0 \) echo train

- 3DFSE — like 2D except
  wind/unwind added to thick
  Slice select (w/ emskers on 180°)

N.B.: only one read after, subsequent 180° resets
  from right to left

N.B. all those 180° pulses deposit a lot of
  RF power:
  \( 90° + 180° = 45° \text{ power } 30° \)
MULTI-SLAB 3DFSE, PROBLEMS

- echoes die out quickly $\propto e^{-t/\tau}$
- since echoes are $90^\circ$ limited to $<30^\circ$, can't fill 3-D k-space in a reasonable time
- SAR constraint $SAR \propto B_0^2 \Theta^2 A f$
  $\Rightarrow 180^\circ$ pulses deposit 4-6X power of $90^\circ$
- "multi-slab" is halfway between slices and single-slab
- problem at slice boundaries — esp. movement
- multilab requires slice selective RF pulses $\Rightarrow$ longer than non-selective 'hard' pulses

$\Rightarrow$ echo train e.g. 20
$\Rightarrow$ etc to fill 3D k-space
$G_z$ is "partition"
$G_y$ is "phase encode"
$G_x$ readout needs no pre-wind since $180^\circ$ does it

$T_{eff}$ is $90^\circ$ to echo thru center of k-space

$4\text{ms} \text{RO}$

hard to get under 8 msec inter-echo spacing

limits speed of covering k-space
SINGLE-SLAB 3D FSE


- regular FSE (180° pulse train)

- sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)

  - this "storage" in Z-axis preserves magnetization for longer time

  - smaller flip angles allow much longer echo trains

  - enough to collect whole plane of 3-D k-space

- different than hyper echoes (not symmetric)

- contrast must consider STE

\[
SE = \sin \alpha, \sin^2 \alpha \frac{e^{-2\pi T_2}}{T_2}
\]

\[
STE = \frac{1}{2} \sin \alpha \sin^2 \alpha \sin^3 \alpha e^{-\pi/12} e^{-2\pi/12}
\]

- single-slab 3D FSE pulse seq.

variable flip angle (<1 msec)

hard (non-selective) pulse not 180°

long T_2 (T2)

short T_2 (PD)

kx

ky

kz

26

16

6

5

train 1

don't collect

train 2

don't collect

train R

don't collect

NB: time to scan k-space is \( \propto S^2 \)

apparent contrast time b/c of "storage"

(e.g. T_2 eff = 585 ms looks like FSE T_2 = 140 ms)
FAST GRADIENT ECHO (GRASS | FLASH | FISP | SPGR | MPRAGE)

- Small tip so TR can be greatly reduced (e.g. 10 msec, less than T2)
- 'Leftover' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

STEADY-STATE COHERENT (GRASS, FISP)

- Unwind phase from phase-encode M_r before next pulse (here because TR<TE)
- Unwind read gradient, too

STEADY-STATE SPOILED (SPGR, FLASH)

- Spoil with random gradient (but this still allows some Ax refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T1-weighted)

NON-STEADY STATE, MAGNETIZATION-PREP

- Preparation pulse → Strong T1-weighting
- Contrast varies in spatial-frequency-dependent way

180° x (MP-RAGE)

- Longitudinal mag. not affect much by low angle pulses

Record k_y = 0 here
QUANTITATIVE T1 - HELMS 2-FIiP ANGLE METHOD

- start w/ gradient echo signal e.g., dropping T2 decay \( e^{-\Delta T / T_2} \)

\[
S_{\text{Ernst}} = A \cdot \sin \alpha \cdot \frac{1 - e^{-TR/T_1}}{1 - \cos \alpha \cdot e^{-TR/T_1}}
\]

Ernst eq. (max: \( \cos \alpha e = e^{-TR/T_1} \))

- Simplify/linearize/estimate

\( TR \ll T_1 \)

linear approx. of exponentials

Taylor expansion, simplification of \( \sin, \cos \), drop small term

Helms et al. (2006)

\[
S \approx A \cdot \frac{TR/T_1}{\alpha^2/2 + TR/T_1}
\]

- Solve for TD and A (protm-density) given signals from 2 diff flip angles

\( \max \alpha^2/2 = TR/T_1 \)

- Tiny error for flip \( \leq 15^\circ \)

- Solve for T1

\[
T_1 = 2TR \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2/\alpha_2 - S_1/\alpha_1}
\]

- Problem: flip angle varies \( \pm \) at 3T (e.g., 25%) from nominal/requested (e.g., flip series)

- Collect spin-echo and stimulated echo (SE)

\[
S_{\text{SE}} = k \cdot \sin^3 \alpha \cdot e^{-\Delta T / T_2}
\]

\[
S_{\text{STM}} = k' \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-\Delta T / T_2}
\]

\[
\alpha = \arccos \left( \frac{S_{\text{STM}} \cdot e^{-\Delta T / T_2}}{S_{\text{SE}}} \right)
\]

Jiru & Klose (2006)
ECHO PLANAR IMAGING, EPI

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

RF 90° pulse sequence

- Since there is only one RF pulse per slice, spins never get reset to all the same (= zero freq, center of k-space).

Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the G_y "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.

k-space traversal

Individual time points of recording of one echo
**SPIN ECHO EPI**

Why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting.

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing.

- The excess of deoxyhemoglobin (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect.

- Spin echo corrects/cancels static $T_2^*$ ($T_2'$) dephasing, incl. deoxy.

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing.

- Diffusion exposes spins to different fields (reducing gradient echo dephasing).

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels.

- For $TE \approx 100\, ms$, spins diffuse 10's of μm, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time.

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion is less likely to expose spin to different fields here).

- This argument only works for extravascular spins—intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells.

- Measure intravascular with bipolar pulse which kills signal in faster moving blood in moderate and larger vessels.

- Over half of SE-BOLD at LST is venous...
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence
- "Spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space
- "asymmetric spin-echo EPI" arranges for the spin echo to occur 2 msec before the gradient echo, which gives more T2*-weighting (for ky=0 echo)

the 180°-pulse rephasing reduces the T2* signal, which is why the partially rephased asymmetric spin echo has been more commonly used

- at higher fields, spin echo EPI is more promising
  - signal to noise is higher so we can take spin echo hit
  - contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording
- Coils fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff. fall-offs

> but what does this look like in k-space?

- Slow variation in RF-field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space

  ↴(N.B. not addition!)

- To see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space—at all spatial frequencies!!

- Simple example w/ "brain" consisting of one spatial freq.

  ![Image](image.png)

  Image domain

  Spatial freq. domain

  k-space

  → FT

  * (convolve)

  So can undersample k-space!!

  N.B. inverse FT of k-space data "smeared" in spat freq.

  Space is sharp image w/ fall-off (not blurred image.)

  "Smear" means normally orthogonal spat. freq's leak to adj. freqs.

  **GRAPPA** — construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center across multi coils

  **SENSE** — general linear inverse approach

  N.B.: neither would work unless normally orthog. spat. freqs. blurred!
- excite multiple slices at once
- function of $G_2$ blips is to shift slices in $G_y$ direction

- this occurs because for given slice, a phase pedestal is added to the entire slice

\[ \text{[N.B.: different from $B_0$ defect-induced incremented phase errors]} \]

- problem w/ all up $G_2$ blips $\rightarrow$ phase error builds up

**Trick #1**
- start w/ 2 slices, one at $z=0$, other above

\[ \Rightarrow \text{if } \pi (180^\circ) \text{ phase shift used, blip up/down same! (no effect at } z=0) \]

\[ \Rightarrow \text{i.e., move top or bottom replica} \]

**Trick #2**
- for multiple slices not all at $z=0$, phase no longer same for even/odd

\[ \Rightarrow \text{but can add phase to equilibrate to } k\text{-space before recon.} \]

**Trick #3**
- for more than 2 slices:

\[ \text{1st even odd even odd etc.} \]
MULTI BAND/BLIPPED CARI

- relation between leave-one-out aliasing and nominally fully-sampled SMS
  - leave alternate lines out wraps image
  - SENSE/GRAFFA to fix blc coil view swears k-space data
  - nominally, w/ SMS we record every line of k-space
  - but equivalent to leave alternate out blc our multi-slice FOV was not big enough

- slice - GRAFFA
  - reg GRAFFA -> recon missing lines
  - slice GRAFFA -> recon multiple k-spaces
    - for each overlapped slices
      - by training on fully-sampled data
      - at beginning of scan

- interslice "leakage block"
  - when training GRAFFA kernel on fully-sampled data,
    - also minimize interslice leakage (split-slice-GRAFFA)
  - can also do regular GRAFFA on top of this
    - reason: for diffusion, loss in S/N from undersample
      - cancelled by shorter TE readout
      - gain from reduced image distortion from shorter readout
ECHO-VOLUME IMAGING EVI

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

- entire k-space must be filled before 3D image is reconstructed

- main issue is movement artifact since data assembled from many shots over several secs

- breathing-induced B0 problems in different partitions may cause blur
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less
  gradient power required than w/ trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform
  from resonant circuit w/ non-uniform sampling to get constant $\Delta k_x$

- sinusoids in both $G_x$ and $G_y$ give spiral k-space trajectory

- constant angular velocity goes too fast at large $k_x$, $k_y$
- constant linear velocity better but impractical near $k_x=0$, $k_y=0$
- compromise: start constant angular, end constant linear

**Constant angular velocity**

$w(t) = w_0 t$

$k(t) = A t e^{i w_0 t}$

$G(t) = \frac{1}{i} \frac{d}{dt} k(t)$

$= A e^{i w_0 t} + i A w_0 e^{i w_0 t}$

$G_x(t) = A \cos w_0 t - A t w_0 \sin w_0 t$

$G_y(t) = A \sin w_0 t + A t w_0 \cos w_0 t$

**Constant linear velocity**

$w(t) = w_0 T T$

$k(t) = A T e^{i w_0 T T}$

$G(t) = \frac{1}{i} \frac{d}{dt} k(t)$

$= \frac{A}{2 T} e^{i w_0 T T} + \frac{A}{2} w_0 e^{i w_0 T T}$

$G_x(t) = \frac{A}{2 T} \cos w_0 T T + \frac{A}{2} w_0 \cos w_0 T T$

$G_y(t) = \frac{A}{2 T} \sin w_0 T T + \frac{A}{2} w_0 \sin w_0 T T$

$G_x$

$G_y$
SPIRAL 3D IR FSE  (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)

- All echoes after 90° derive from mag w/ same T1 contrast (vs. non-steady-state)
- Possible to present sign
- High uniform contrast, but lots of waiting (T1), high BW

RF

180°(prep1) TI ~ 700 msec  //  90°

Gz

Gy

Gx

Sig.

180° 180° 180° 180°(prep2)

X x 16

Loop order

3D k-space

(“stack of spirals”)

Spiral interleaves

k_z interleaves

k_z echoes

 echoes (after one 90°)
**PHASE ERRORS & ECHO-CENTERING ERRORS**

- Anything that causes a deviation of the $B_2$ field strength from the expected value $(B_{0,z} + G_{x,z} x + G_{y,z} y + G_{z,z} z)$ changes precession frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

**Fourier shift theorem**

- Phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x-x_0) = \int e^{-i2\pi k_x x} S(k_x) e^{i2\pi k_x x} \, dk_x$$

- Correct with shimming and $B_0$ mapping/phase unwrapping before reconstruction.

**Echo Centering Error**

- If realignment of all spins ($k_x = k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted.

- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction, which causes magnitude image unchanged.

**Fourier freq. shift theorem**

- Frequency shift in freq. domain causes phase shift in spatial frequency space.

$$e^{i2\pi k_x x} I(x) = \int S(k_x - k_{x_0}) e^{-i2\pi k_x x} \, dk_x$$
FAST SCAN ARTIFACTS  

EPI vs. Spiral

**EPI**
- \(G_x\) readout gradient strong \(\rightarrow\) field defects smaller percentage less deformation of \(k_x\) (vertical stripe components)
- \(G_y\) "blips" are weak and total \(G_y\) readout time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the \(y\)-direction, for example, maps and unmaps phase as a function of \(x\)-position
- but \(G_x\) big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates

The lack of blurring has lead to a preference for EPI, despite the substantial image shifts

**Spiral**
- with center-out spirals phase errors accumulate in a radial direction
- thus, higher spatial frequencies have more error (=more shearing)
- for spurious \(x\)-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq.

- for defects with more complex contours in the \(y\)-direction (than linear, as above) the vertical shifts (for EPI) will vary with \(y\)-position, and may result in signals from different \(y\)-positions being reconstructed on top of each other
**IMAGE-SPACE VIEW OF LOCALIZED B\(\Phi\) DEFECT, EFFECT ON RECON**

- Localized B\(\Phi\) defects often arise from air pockets embedded in tissue.
  - Air in middle/outer ear \(\rightarrow\) indentation in inferior temporal lobe.
  - Air in endolymphatic epithelium \(\rightarrow\) orbitofrontal d/c, ant. thal. compression.

**Collect one data (k-space) point**

\[
\begin{align*}
\text{localized } & \quad \text{brain structure}\nonumber \\
\text{4 cycles of } & \quad \text{sampled with distorted stripes}\nonumber \\
\text{phase in 4-dir (3-gons)} & \quad \text{complex multiply}\nonumber \\
\text{(g-positions)} & \quad (=\text{correlate sines with brain})\nonumber \\
\end{align*}
\]

**Reconstruction**

\[
\text{from distorted data points} \quad \text{...} \quad + \quad \begin{bmatrix} \text{amplitude} \\ \text{and phase} \end{bmatrix} \quad \text{of this component} \\
\text{undistorted} \quad \text{[same for 5 cycles]} \\
\text{stripes used by} \quad \text{[same]} \\
\text{inverse FFT} \quad \text{...} \quad = \quad \text{brain image}
\]

**N.B.:** Image phase only occurs if shift is not sampled w/ successively later echoes (see next page)

**Close-up of distorted phase stripes (one cycle)**

- Same defect makes leftward dent in vertical phase stripes.
- Spatial information can be lost when continuous changes in phase are flattened by B\(\Phi\) defect.
- Shifts can pile multiple pixels on top of each other into one bright pixel.

- Local estimates of B\(\Phi\) needed to correct images:
  1) Fieldmap method: < Multiple TEs to est. local B\(\Phi\) from \(\phi / T_E\) slope.
  2) Point-spread-function: < Extra phase encode to estimate PSF (should be E-function)
     \quad deconvolve distorted image in phase-encode direction.
LOCALIZED BΦ DEFECT, EFFECT ON RECON

- When local BΦ defect distorts image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- If each successive k-space recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space.

- A k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- However, with w/EPI, static BΦ defect causes more and more local displacement of image phase stripes for each additional k-space line.

- That is, later lines have greater spat. freq. offset.
- Effectively stretches k-space in ky direction.
- Same num samples to higher spatial freq.
- Shrinks FOV (squishes voxels — see FOV page)

- When image is reconstructed, region with local BΦ defect shifted oppositely.

- Thus, local shift effect due to combination of 3 things:
  1) Static local ΔBΦ defect
  2) Successive increases in phase error for successive spat. freq. measurements during long EPI readout
  3) Small size of ky phase encode blips relative to BΦ defect (much less of this effect in freq. encode direction)

- Respiration (which affect BΦ) in 3D FLASH might cause similar effect within k2 partition (if successive spat. freqs.)
GRADIENT NON-LINEARITIES

- Ideally, the Gx, Gy, and Gz gradient coils attempt to impress a linear variation onto to Z-component of the B field — Bz — in the x, y, and z directions.
- In practice, gradient coils are non-linear (e.g., printed-circuit-like).
- Non-linearities are worse in smaller coils, but also in higher-performance coils, designed for post-processing correction of distortions.
- Non-linearities result in phase errors, which result in 3-D image distortion.

- A non-linear slice select gradient will excite a curved slice.
- Non-linear phase and frequency encode gradients will distort in-plane features.
- Some scanners correct these differently: 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!).
- This can result in errors approaching 1 cm in function.
- Different coil designs have different directions of distortion (!).
- The assumption of non-Maxwellian gradients results in additional phase errors.
- These can also be corrected since the Bx and By components are known.

That is, the assumption that gradients cause no field in the Bx and By direction.

These effects do not build up over time in phase encode direction since they only occur when gradients are turned on.
SHIMMING AND $B_0$-MAPPING

- Passive iron shims inserted to flatten $B_0$ field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the $B_0$ field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc) (= several hundred ppm)

Linear shim coils impose gradients in x, y, and z
Higher order shims impose higher order spherical harmonic field components (e.g. $z^2$)

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the $B_0$ field

Local resonance offsets caused by $B_0$ defects estimated from images
- e.g., sample phase at multiple echo times

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  - This only corrects spatially gradual field defects
  - Local defects due to air in sinuses much higher order than shims

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to unwarp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)
Navigator Echoes

- 1D navigator
  - B₀ drift problem
    - slow up/down drifts in B₀ continuously occur
    - a pedestal in B₀ is pedestal in phase (not gradient) which causes spatial shift (Fourier shift theorem)
    - in EPI, mainly affects phase-encode dir b/c small slip in reading
    - result is successive volumes drift in phase encode dir
  - Gradient balance problem
    - unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space causing N/2 (Nyquist) ghosting
    - another phase error

- 3D navigator: collect 3D sphere in k-space
  - rotation of object → rotation of k-space amplitude pattern
  - translation of object → phase shift of k-space phase (Fourier shift)
  - sample at sufficient radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - do N,S hemispheres separately (less T₂*, cancel EPI-like error accumulation)

Welch et al. (2002) MRM

\[ z(n) = \frac{2n - N - 1}{N} \]

\[ y(n) = \cos(\sqrt{N} \pi \sin^{-1} z(n)) T_1 - z^2(n) \]

\[ x(n) = \sin(\sqrt{N} \pi \sin^{-1} z(n)) T_1 - z^2(n) \]

(skip poles - slow rate too high)

- can be used for prospective motion correction (rotate, translate w/ gradients)
- better estimate, because of speed, than full TR & EPI images (24 ms vs. 2-4 sec)
- may need to smooth rot, transl estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way.
  - variations can be used (cf. GRAPPA, SENSE) and/or corrected

- transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
  - potentially worse (why local transmit is still in progress)
  - usu. fixed by using a large transmit coil (e.g. body coil)

- RF penetration at higher fields (= higher RF frequencies)
  - is less uniform:
    1) decreased RF wavelength (closer to size of head) at higher freq.
    2) increased permittivity (ε) and conductivity (σ) at higher field

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)

- different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)

- record lo-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s)
- divide small coil body coil at each voxel to determine receive field
- use receive field to normalize main image(s)
  [ see also: qT1, MP2RAGE, T1/T2 ]
**Diffusion-Weighted Imaging**

- Spins acquire phase during first $\delta t$
- If spins diffuse (= move) along gradient by time $T$, signal is lost because negative $\delta t$ doesn't re-phase
- Attenuation: $A(D) = \frac{S_{\text{no DW}}}{S_{\text{DW}}} = e^{-bD}$
  where $b = \gamma^2 G^2 \delta t^2 (T - \delta t/3)$

- Apparent diffusion coefficient map
- To get large $b$, need $G \uparrow \delta t \uparrow$ (need big $G$’s)
  Long $\delta t$ gives spurious $T_2$-weighting
  Can use stimulated echoes:
  $90^\circ$ RF — $\delta t$ — $90^\circ$ RF — $\delta t$ — transverse
  Diffusion lobes 1 — park in longitudinal back to transverse diffusion lobes 2

1) Anisotropic Diffusion (Gaussian)

- Measure $D$ along multiple axes
- Have to measure tensor, not scalar
  Even for determining one primary direction
  $D = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix}$
  $\mathbf{D} = \mathbf{u}^T \mathbf{D} \mathbf{u}$
- Since $D$ is symmetric, only need 6 measurements
  $\Rightarrow$ i.e., paired diffusion lobes applied along 6 axes
- Without 3rd number ($\theta$) $x$ & $y$ projections same

2) Length Scale by multiple $b$-values
- Fit line to semi-log signal as function of $b$
- If not straight line: multi-exponential, e.g., $S = A_1 e^{-bD_1} + A_2 e^{-bD_2}$
- Fiber tract mapping
- An ill-posed prob.
  E.g., constraint both ends and central point (!!)
- Need vis. areas test
**PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ**

- **Spin-echo 'Stejskal-Tanner' EPI seq.** (standard on scanners)
  
  - Allows longer TE
  
  - Flips $M_z$ so rephase gradient has same sign as dephase

- **Eddy-currents** are long time-constant currents in metal of scanner that distort B field → spatial image distortion

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/partic. time constants)
  
  - Also, keep crushers orthogonal to diffusion-encoding gradients

- Nagy et al. (2014) MRM

  - $\mathbf{Y}_{\text{TRSE}} = 0 = \mathbf{Y}_1 - \mathbf{Y}_2 - \mathbf{Y}_3 + \mathbf{Y}_4$

  - Phase dispersion (6 echo)

  - Twice-refocused Spin-echo (for center k-space)
PERFUSION - ARTERIAL SPIN LABEL

- Basic idea: Tag blood below area of interest, collect control & tagged image, assume directional input flow.

Continuous ASL (CASL) - Continuously tag a plane, gradient on, blood gets adiabatically inverted as it passes through location w/coned resonant freq.

Pulsed ASL (PASL) - e.g., EPISTAR, FAIR, PINEC, QUIPPS II
   Tag block of tissue below slice(s)

- Small diffs between control and tag (~1%)
   Requires accurate balancing of control & tag images

- Contrast problems:
  - Transit delays -> biggest confounding factor
  - Relaxation rate diffs
  - Venous clearance (vs. microspheres, which get stuck)

Solutions for quantitative
- Insert delay so all spins arrive into low velocity capillaries
- Kill end of tag to reduce spatial variation of tag

QUIPPS II - Quantitative perfusion

1) Pre-saturate spins in target slices
2) Tag - 180° pulse below slices
   Control - 180° pulse above slices (to control off-resonance)
3) Saturate tagged block to end tag (TI)
   Both tag and control can use train of thin slices pulses at top of tag band
4) EPI of spiral images of target slices (TI)
   Image most distal slice fast to cancel delays
   Fast between slice so imaging excitation don't get interpreted as flow

\[ \Delta M = \text{flow} \times \left[ 2M_0 \text{TI} , e^{-TI_2/T1a} \right] \]

Can extract flow and BOLD adjacent substrations minimize movement artifact

1) Alternate tag and control, GRE TE = 30 ms
   Control = tag = flow
   Control + tag = BOLD

2) Dual echo spiral
   k = 0 early => high S/N flow
   TE = 30 ms => BOLD
OFF RESONANCE EXCITATION

- Main idea: examine evolution of $\mathbf{M}$ vector in rotating coord syst set to "off-resonance" $\mathbf{B}_1$ field freq (Wrf), not Larmor freq of $\mathbf{M}$ ($\omega_0$).

- Normally, if rotating coord syst freq set to Larmor freq ($\omega_{\text{rf}} = \omega_0$), an actually precessing $\mathbf{M}$ will be stationary (ignoring decay) \textrightarrow{implies effective $B_z = 0$ in rotating coord syst}.

- Now, move $\mathbf{M}$ to rotating coord syst at $\mathbf{B}_1$ freq lower than $\omega_0$ (assume $\mathbf{B}_1 \gg \mathbf{B}_z = 0$): existing $\mathbf{M}$ will now appear to precess around $z$-axis.

  N.B.: this is precession in already rotating coordinate system!

- $\Delta \omega_0 = \omega_0 - \omega_{\text{rf}}$
  - freq of precession in rotating coord syst
  - Larmor rotation freq of $\mathbf{M}$ at $\omega_0$ ("incorrectly set" rotating coord syst freq).

- Thus, viewing $\mathbf{M}$ vector in off-resonance rotating coord syst makes it look like additional $\mathbf{B}_z$ field is causing "extra" precession.

- "Extra" $\mathbf{B}_z$ component is proportional to $\Delta \omega_0$ offset.
  - Can be pos or neg: too low $\rightarrow$ pos $\mathbf{B}_z$
  - Extra $\mathbf{B}_z$ adds to $\mathbf{B}_1$ resulting in precession around tilted axis: $\mathbf{B}_{\text{eff}}$ (effective).

- Extra $z$-gradient can have same effect on $\Delta \omega_0$ (changes $\omega_0$ instead of changing $\omega_{\text{rf}}$).

- $\mathbf{B}_{\text{eff}} = \left( \frac{\Delta \omega_0}{\gamma} \right) \mathbf{k} + B_{\text{extra}} \mathbf{k} + B_1 \mathbf{i}$
  - Effective $\mathbf{B}$ in rotating frame set to $\mathbf{B}_1$ freq.
  - Apparent "extra" $\mathbf{B}_z$ from Larmor-$\mathbf{B}_1$ freq mismatch (pos or neg) (if on-res. $\rightarrow$ 0).
  - Optional $z$-gradient (pos or neg) (additional source of $\mathbf{B}_{\text{eff}}$ tilt).
  - Transverse RF stim (here, around x-axis).

adiabatic RF pulse $\approx$ flow-driven CASL tag

RF: sweep freq $\omega_0$: constant (for given slice)
RF: const freq $\omega_0$: sweeps because spins flow along gradient direction.
**SPECTROSCOPY + IMAGE**

- **chemical shift**: small displacement resonant freq due to shielding of target nucleus (e.g., H) by surrounding electron orbitals

  - e.g., acetic acid:
  
  ![Chemical structure of acetic acid]

  - oxygen attracts electron so less shielding of target nucleus
  - 3 of these H's (more shielded)
  - 1 of these H's (less shielded)

- how we get chemical shift spectrum:

  - RF stim
  - Larmor oscillations are multiplied by center freq to obtain Δf (not MHz high freq)

  - data before FT is a series of time-domain samples of the mix of shifted-freq offsets

  - FT turns data into "shift spectrum"

- **Pulse Sequence**

  - since we are already using phase (freq) encoding for space, we need one "extra dimension" w/ all gradients OFF!

  - use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal & FT-it like chemists do!

- N.B.: opposite "direction" of FTs!
**PHASE-ENCODED STIMULUS & ANALYSIS**

**Map Polar Angle**

**Map Frequency**

**Map Eccentricity**

**Map Prox/Distal Axis, Road Maps**

**Periodic Stimuli (Phase-encoded)** - e.g., 8 cycles at 64 sec/cycle

**Calculate Significance**
- Ratio between amplitude at stimulus frequency (=signal) and average of amplitudes at other frequencies (=noise)
- Ignore harmonics, low freq (=movement)

**Smooth**
- Vector average of complex significance \((A, \phi)\) with that at nearest neighbor surface points

**Display**
- Plot phase using hue and saturation to indicate significance

**Delay Correction**
- Record responses to opposite directions of stimulus (ccw/cw, inflow, up/down)
- Vector average after reversing angle of one
  - Penalizes inconsistent more than just avg of angles

**Typically 0.5 - 5% amplitude**

**Strongly Periodically Activated Single Voxel Time Course**

**Remove Constant (avg) and Linear Trend**

**Real**

**Imaginary**

**FFT, convert to \(A, \phi\)**

**Freq = total TRs/2**

**Reversed CCW Vector Average**

**CW Significance**

**Complex CCW Significance**
**CONVOLUTION**

\[ f(x) = g(x) \ast h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x - z) \, dz \]

- **Definition of Convolution**

- How to calculate one term
- Sum across all \( z \) to get the value of the convolution at \( x \)
- Move kernel to calculate next \( x \)

Why we reverse

**Impulse response function (HRF)**

**Impulses (ERP design)**

How to calculate convolution for this time point (only 3 terms in integral - all other zero)
**GENERAL LINEAR MODEL**

\[
\hat{y} = \hat{X}\hat{h} + \overline{S}\hat{b} + \hat{n}
\]

- **Goal is to solve for the hemodynamic response functions, \( h \)**

- **Data = design * HDR + drift * weights + noise**

\[
\begin{bmatrix}
\text{texp} \\
\text{themo} \\
\text{data}
\end{bmatrix}
= \begin{bmatrix}
\text{h} \\
\text{X} \\
\text{S}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{texp} \\
\text{themo} \\
\text{data}
\end{bmatrix}
= \begin{bmatrix}
\text{h} \\
\text{X} \\
\text{S}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{texp} \\
\text{themo} \\
\text{data}
\end{bmatrix}
= \begin{bmatrix}
\text{h} \\
\text{X} \\
\text{S}
\end{bmatrix}
\]

- **Maximum likelihood estimate**

1) Assume white noise, solve for \( \hat{h} \)

2) \( \hat{h} = (X^T P_s^T X)^{-1} X^T P_s^T y \) where \( P_s^T = I - S(S^T S)^{-1} S^T \)

   or

   \( = (X_{\perp}^T X_{\perp})^{-1} X_{\perp}^T y \) where \( X_{\perp} = P_s \times X \)

3) **Significance** (how to construct F-ratio)

   \[
   F = \frac{N-K-L}{K} \begin{bmatrix}
   y^T (P_{xs} - P_s) y \\
   y^T (I - P_{xs}) y
   \end{bmatrix}
   \]

   \( \Rightarrow \) design matrix \( \{\text{h} / \text{noise} \} \) geometric interp
**GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL**

- With no nuisance functions \( S \), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

\[
\hat{y} = Xh + \hat{e} \quad \Rightarrow \quad Xh = P_y \hat{y}
\]

Projection matrix \( P_y \) operates on \( \hat{y} \) to give projection of data into experiment space, \( X \).

- When nuisance functions, \( S \), are considered, problem: \( S \) may not be orthogonal to \( X \).

\[
\begin{align*}
&\text{Orthogonal projection} \\
&\text{Oblique projection}
\end{align*}
\]

For example: linear trend not orthogonal to std. block design.

**Geometric Picture**

(Liu et al. 2001, Neuroimage)

**XS:** space of data modeled by all reference and nuisance

**Exy:** oblique projection onto nuisance (\( Exy \))

**PXY:** orthogonal projection onto nuisance (\( P_{XY} \))

**Error (e):** not explained by reference and nuisance (\( F \) denoted)

\[
(I - P_{XY})y
\]

How much more of data you can explain by adding reference functions (\( F \) numerator)

\[
(P_{XY} - P_{S})y
\]

Same as projection onto reference only in special case where \( S \perp X \).
1) MNI to Talairach → generates 4x4 matrix
   - make average brain target (blurry)
   - blur target (further), blur single brain (a lot), gradient descent on Xcorr
   - repeat w/ less blurring of avg target and current brain
   - problems: variable neck cut-off
equal to but much better than standard!
   - only 2 points near center of brain!
   - fit to bounding box

2) Intensity Normalization (output: "T1")
   - histogram of pixel values in 10 mm thick T1R slices
   - smooth histogram
   - peak find to get initial estimate of white matter
   - discard outlier peaks across slices
   - fit splines to peaks across slices
   - interpolated scaling factor 1 to T1R
   - scale each pixel so WM peak is 110
   - refine estimate to interpolate in 3D
   - find points in 5x5x5 within 10% of WM, get near scale for them
   - build Voronoi to interpolate scales unset above
   - soap-bubble-smooth Voronoi boundaries (3 iterations)
   - re-scale each voxel

3) Skull Stripping (output: "brain")
   - "shrink-wrap" algorithm
   - start with ellipsoidal template
   - minimize brain penetration and curvature
   - curvature: spring force (from center-to-neighbor vect sum)
   - brain penetration
   - apply force along surface normal that prevents surface from entering gray matter
- implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something")
- more formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrink wrap update eg. (skull strip, original Dale & Sereno surface refinement)

\[
\mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t)
\]

\[
\mathbf{F}_{\text{smooth}} = \lambda_{\text{smooth}} \sum_{\text{neigh}} \left( \mathbf{I} - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T \right) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})
\]

\[
+ \lambda_{\text{normal}} \sum_{\text{neigh}} \left( \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T \right) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) - \frac{1}{\# \text{vertices}} \sum_{\text{neigh}} \sum_{\text{neigh}} (\mathbf{n}_v \mathbf{n}_v^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_v)
\]

\[
\mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \max_{d} \left[ 0, \tanh \left[ \frac{\mathbf{I} (\mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}}) - I_{\text{threshold}}}{\max_{\text{force}}} \right] \right]
\]

Snapshot of surface and "core sample" from one vertex
4) Non-isotropic filtering (output: "win") — "floss" and "speculate"
   - preliminary hard thresholds: output
   - find ambiguous/boundary voxels
     \[ 20\% \text{ or more of } 26 \text{ immediate neighbors different} \]
   - find plane of least variance
     \[ \text{for each direction (from icosahedral superfine tessellation)} \]
     \[ \text{consider } 5 \times 5 \times 5 \text{ volume around 1 voxel} \]
     \[ \text{find plane of least variance in this hemisphere} \]
     \[ \text{medium filter w/ hysteresis} \]
     \[ \Rightarrow \text{if } 60\% \text{ of within-slab differ, reverse classification} \]
     \[ \Rightarrow \text{"flosses" sulci without blurring} \]

5) Find cutting planes
   - midbrain
   - callosum, to separate hemispheres (SAG)
   - midbrain, to avoid fill into cerebellum (Talairach)
     \[ \text{Talairach to start; fill WM in SAG or HDR till min area} \]

6) Region-growing to define connected parts (output: "filled")
   - inside-out, outside-in, inside-out — for each hemisphere

   - up/down cycles within each plane
   - plane-by-plane

   - "wormhole filter" (3x3x3 = center + 26)
     \[ \Rightarrow \text{fill (untamed) voxel if 66\% neighbors differ} \]
     \[ \Rightarrow \text{eliminates structures within, 1-D structure} \]
7) Surface Tessellation (output: rh.orig, lh.orig)

- Find filled voxels bordering unfilled
- Make ordered list of neighboring vertices
  ⇒ so cross-products oriented properly

- Long list of values associated with each numbered vertex
  e.g. position (orig, morphed)
  area (orig, morphed)
  curvature (intrinsic, Gaussian)
  "Suclusness" (summed 1 movement during unfolding)
  cortical thickness
  fMRI data
  EEG/MEG dipole strength

- Separate fMRI data set must be aligned, sampled
  fMRI voxels larger
  Sample at each surface vertex
  nearest-neighbor "soap bubble" smoothing
to interpolate data onto hi-res mesh

- Some quantities only well-defined on surface
  ⇒ gradient of magnitude of cortical map measure (e.g., eccentricity)
- Smoothing/inflation/WM, pial done as derivative of energy functional

$$J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}}$$

- Total scalar tangential error (fixed by redistributing vertices)
- Normal scalar curvature error (fixed by reducing curvature)
- Image scalar image error (fixed by moving towards target image value)

$$J_{\text{normal}} = \frac{1}{2 \# \text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{n}_{\text{center}} \cdot \left( \mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}} \right) \right]^2$$

- Across all vertices
- \(1/2\) so no coefficient on derivative
- Across vertices of one vertex
- Vertex unit normal
- Project vector from current center to any neighbor (position vector difference)
- Vector from current center to any neighbor (position vector difference)
- Project onto tangent plane
- X-direction in tangent plane
- Y-direction in tangent plane

$$J_{\text{tangential}} = \frac{1}{2 \# \text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{t}^x_{\text{center}} \cdot \left( \mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}} \right) \right]^2 + \left[ \mathbf{t}^y_{\text{center}} \cdot \left( \mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}} \right) \right]^2$$

- Across all vertices
- \(1/2\) so no coefficient on derivative
- Across vertices of one vertex
- Vertex unit normal
- Project vector to neighbor onto x & y
- X-direction in tangent plane
- Y-direction in tangent plane

$$J_{\text{image}} = \frac{1}{2 \# \text{vert}} \sum_{\text{centers}} \left[ I_{\text{center}} - I_{\text{image}} \left( \mathbf{r}_{\text{center}} \right) \right]^2$$

- Across all vertices
- \(1/2\) so no coefficient on derivative
- Brightness at current location
- Image for WM: mean of voxels labeled WM in 5 mm neighborhood
- Image for pia: global - small num for CSF-like

- Take directional derivative of energy functional (to find steepest uphill)
- Move each vertex in the opposite (negative) direction w/ self-intersect test

$$- \frac{\partial J}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{image}} \left( \mathbf{r}_{\text{center}} \right) - I_{\text{center}} \right] \nabla I_{\text{image}} \left( \mathbf{r}_{\text{center}} \right)$$

- Go in the opposite direction (vector) of largest scalar error for one vertex
- Sum over neighbors
- Normal scalar curvature error (fixed by reducing curvature)
- Unit normal vector scaled by dot product
- X-component of tangential
- Y-component of tangential

- Note: eq. 9 in Lobo, Fischl & Sapiro different — and incorrect!
SULCUS-BASED CROSS-SUB. ALIGN

- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"
- Add term to energy function: "sulcus-ness" error: \((S_{\text{ent}} - S_{\text{targ}})^2\)
- Bootstrap: Morph to one brain, make avg target, remorph to avg target

Smooth WM \rightarrow \text{inflated} \rightarrow \text{sphere} \rightarrow \text{registered sphere}

Sub_1 \rightarrow \text{inflate} \rightarrow \text{sphere} \rightarrow \text{morph} \rightarrow \text{interpolate}

Sub_2 \rightarrow \text{...} \rightarrow \text{...} \rightarrow \text{...} \rightarrow \text{...} \rightarrow \text{...}

Sub_n \rightarrow \text{...} \rightarrow \text{...} \rightarrow \text{...} \rightarrow \text{...} \rightarrow \text{...}

- Each sub's native surf has diff # vertices
- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)
- Average surface made from folded/inflated avg coords
  - Folded: loses area from sulcal crinkles (ts average "inflated")
  - Inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")
- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

\(\Rightarrow\) N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)