MAGNET HARDWARE

- $B_0$ field from superconducting magnet
- RF transmit/receive
- gradient coils

$B_0 \rightarrow z$ (longitudinal)
$B_1 \rightarrow x, y$ (transverse)

(1) $B_0$ field

superconducting coils in liquid helium (no power required after current injected to bring up field using induction)

$1T = 10,000$ Gauss
$rac{1T}{2\pi} = 42.6$ MHz/T

(2) body gradient coils

(3) RF transmit body coil

(4) RF receive-only head coils

shim coil also embedded in here (not shown)

RF transmitter (10 kW)
RF receiver

circularly polarized $B_1$ field rotating 90 degrees to $B_0$ at Larmor frequency (B1 is several orders of magnitude smaller than $B_0$)

three one million watt amplifiers to add ramps to $B_0$ field

usu. water cooled
SPIN & PRECESSION

- nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers)
- moving charge creates magnetic field
  ![classical picture](image)
- current loop from spinning charge (right-hand rule)
- N.B.: classically this would cause EM radiation, spindown
- Stern-Gerlach experiment
  ![experiment](image)
  pass nuclei through strong mag. field → split into just 2 beams

Microscopic picture

<table>
<thead>
<tr>
<th>Strong magnetic field, ( B_0 ) = ( \uparrow )</th>
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<tr>
<td>Strong ( B_0 ) plus oscillating ( B_1 )</td>
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Macroscopic picture

- bulk magnetization \( \equiv 0 \)
- bulk magnetization \( = M_0 \)
- bulk magnetization precesses \( t_1, t_2, t_3 \)

Precession

- distinguish precession (slow) from spin (fast)
- treat classically, like spinning top

\[
2\pi f = \frac{\hbar}{\gamma B_0}
\]

Larmor freq. (eg. 63 MHz)

\[
\gamma \text{ gyromagnetic ratio (eg. 1.5T)}
\]

left-hand rule:

- thumb = \( B_0 \)
- fingers = precession

- bulk equilibrium magnetization (parallel to \( B_0 \))

\[
M_0 = |\vec{M}| = \frac{\gamma^2 \hbar^2}{4K} \tilde{B}_0 \hat{N}_s
\]

where \( I = \frac{1}{2} \)

\[
K = \frac{1}{2} \hbar \frac{1}{3} \]

N.B.: compared to top & gravity:

- frictionless spin
- signed gravity
  - can change precession dir
  - can stick under floor
- neighbor bumping causes decay (\( T_2 \))
**Bloch Equation**

- Time-dependent behavior of $\vec{M}$ in the presence of an applied magnetic field (excitation and relaxation).

For precession: $B = B_0$

For excitation: $B = E + B_0$

Change in $\vec{M}$ vector

- In the Larmor-rotating coordinate system, a tilt who a phase shift in a standard $B_1$ excitation is rotation around $x$-axis.

- Longitudinal and transverse relaxations

\[
\frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_0}{T_1}
\]

\[
\frac{dM_{x,y}(t)}{dt} = -\frac{M_{x,y}(t)}{T_2}
\]

- Solution to equations above: time-dependent free precession e.g.'s

Given initial $M_0, M_p$

- Re-growing from $0$
- Leftover after pulse-decaying

$M_z'(t) = M_z(0) e^{-t/T_1}$

$M_{x,y}'(t) = M_{x,y}(0) e^{-t/T_2}$

Initial condition

$M_0, M_p$ at time instant after pulse

$M_z(1) = 63\% M_0$

$M_{x,y}(0) = 37\% M_{x,y}(0)$

$\approx 100$ msecs
**VECTOR ADD, MULTIPLY**

- Adding vectors is easy
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers

- Multiple ways to multiply vectors: here are 3

**Dot Product**
(= Inner product)
(= "Scaled projection onto")
\[ \vec{c} = \vec{a} \cdot \vec{b} = [b_x b_y b_z][a_x, a_y, a_z] = a_x b_x + a_y b_y + a_z b_z \]
- Scalar
- Generalizes to any \( D \)
- Length of \( \vec{a} \)
- \( \vec{c} = \vec{a} \parallel \vec{b} \parallel \cos \Theta \)
  \( \Rightarrow \) zero if \( \vec{a}, \vec{b} \) orthogonal

**Cross Product**
(= Outer product)
(=Can be generalized: see "Geometric Algebra")
\[ \vec{c} = \vec{a} \times \vec{b} = [0 b_z - b_y, -b_z 0 b_x, b_y - b_x 0][a_x, a_y, a_z] \]
- Vector
- Right hand rule: curl fingers from \( \vec{a} \) to \( \vec{b} \): thumb is \( \vec{c} \)
- Uniquely orthogonal
- Specific to 3D
- \( \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \sin \Theta \)
  \( \Rightarrow \) max if orthogonal

**Complex Multiply**
(See also quaternions, geometric algebra generalization)
\[ \vec{c} = \vec{a} \cdot \vec{b} = [b_x b_y][a_x, a_y] = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]
- Angles add (specific to 2D)
- Magnitudes multiply (like real num)
**Simple Matrix Operations**

**Basic Idea**
- A matrix \( \begin{bmatrix} \text{rotates} \\ \text{scales} \end{bmatrix} \) acts on a vector \( \mathbf{b} = \mathbf{M} \mathbf{a} \)

**3D Example**
\[
\begin{bmatrix}
  b_x \\
  b_y \\
  b_z \\
\end{bmatrix} = \begin{bmatrix}
  M_{11} & M_{12} & M_{13} \\
  M_{21} & M_{22} & M_{23} \\
  M_{31} & M_{32} & M_{33} \\
\end{bmatrix} \begin{bmatrix}
  a_x \\
  a_y \\
  a_z \\
\end{bmatrix}
\]

Add translate (after rotate/scale)
- Commonly used "hack" for aligning, e.g.,
- A 4D matrix \( \begin{bmatrix} \text{rotates/scales} \\ \text{then} \\ \text{translates} \end{bmatrix} \) (4th D = 1)

**N.B.** Have to keep track of order!!
- Rotate/scale then trans \neq\ trans then rot/scale
- Change rot component: untranslate, rot, retranslate

3 special cases (3D): rotate around each major axis without changing length (scale = 1.0)

- Rotate around \( x \)-axis:
\[
\mathbf{R}_x(\alpha) = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \alpha & \sin \alpha \\
  0 & -\sin \alpha & \cos \alpha \\
\end{bmatrix}
\]
- E.g., 90° flip

- Rotate around \( y \)-axis:
\[
\mathbf{R}_y(\alpha) = \begin{bmatrix}
  \cos \alpha & 0 & -\sin \alpha \\
  0 & 1 & 0 \\
  \sin \alpha & 0 & \cos \alpha \\
\end{bmatrix}
\]
- E.g., 180° flip
- To avoid adding 180° phase after 90° flip on \( x \)
- Precession with \( \dot{\phi} \) along \( z \)

General Case
- Rotate around general \( z' \)-axis:
\[
\mathbf{R}_{z'}(\alpha) = \mathbf{R}_z(-\theta) \mathbf{R}_y(-\phi) \mathbf{R}_z(\alpha) \mathbf{R}_y(\phi) \mathbf{R}_z(\theta)
\]

- Quaternion rotations are more efficient
**Solutions to Simple Differential Eq.**

**Diff. Eq.:**
\[ \frac{dM_x y'(t)}{dt} = -\frac{M_x y(t)}{T_2} \]

**Solution:**
\[ M_x y'(t) = M_x y(0_t) \cdot e^{-t/T_2} \]

**Goal:**
1. Find \( e_x \) whose derivative satisfies diff. eq.
2. Also find soln (one of many) that passes thru init condition
   - Since our diff. eq. is: \( \text{derivative of funct.} = \text{const. same funct.} \)
   - Try exponential, since derivative \( (e^r) = e^x \)

**Diff. Eq.:**
\[ M(t) = \frac{1}{T_2} \cdot \frac{dM(t)}{dt} \]

**Deriv. of var:**
\[ M(t) = e^{-t/T_2} \]

**One soln:**
\[ M(t) = e^{-t/T_2} \]

**Take deriv. to check:**
\[ M'(t) = -\frac{1}{T_2} \cdot M(t) \]

**OK!**

Another soln:
\[ M(t) = \text{const.} \cdot e^{-t/T_2} \]

**Take deriv. to check:**
\[ M'(t) = -\frac{1}{T_2} \cdot \text{const.} \cdot e^{-t/T_2} \]

**OK!**

**Initial condition:**
\[ M(t) = (M_x y(0_t)) \cdot e^{-t/T_2} \]

**Information added to soln:**
- Const = \( M_x y(0_t) \)

**Magnetization immediately after RF (B1) ends:**
- Const = \( M_x y(0_t) \)
**BLOCH EQUATIONS**

**Matrix Version**

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi
\]

Differential Eq.: 

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & \gamma B_\phi & 0 \\
-\gamma B_\phi & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Solution: 

\[
\vec{M}(t) = \begin{bmatrix}
M_x(t) \\
M_y(t) \\
M_z(t)
\end{bmatrix} = \begin{bmatrix}
\cos\omega t & \sin\omega t & 0 \\
-\sin\omega t & \cos\omega t & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
M_x(0) \\
M_y(0) \\
M_z(0)
\end{bmatrix} = R_z(\omega t)\vec{M}(0)
\]

Include Relaxation 

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi - \frac{M_x I + M_y J}{T_2} - \frac{(M_z - M_z^0) K}{T_1}
\]

Differential Eq.: 

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{T_2} & \gamma B_\phi & 0 \\
-\gamma B_\phi & -\frac{1}{T_2} & 0 \\
0 & 0 & -\frac{1}{T_2}
\end{bmatrix} \begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
M_z^0 / T_2
\end{bmatrix}
\]

Solution: 

\[
\vec{M}(t) = \begin{bmatrix}
e^{\frac{-t}{T_2}} & 0 & 0 \\
0 & e^{\frac{-t}{T_2}} & 0 \\
0 & 0 & e^{\frac{-t}{T_1}}
\end{bmatrix} \begin{bmatrix}
\cos\omega t & \sin\omega t & 0 \\
-\sin\omega t & \cos\omega t & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
M_x(0) \\
M_y(0) \\
M_z(0)
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
M_z(0)(1-e^{\frac{-t}{T_1}})
\end{bmatrix}
\]
RF FIELD POLARIZATION

- Polarization (change of direction)
- Linearly polarized field
  \[ \vec{B}_1 = B_1 \cdot \cos \omega t \hat{x} \]
  \[ \text{magn. strength \{1, -1\}} \]
- N.B.: \( \vec{B}_1 \) adds to much larger \( \vec{B}_0 \)
  \[ \vec{B}_0 \rightarrow \vec{B}_1 \rightarrow \vec{B}_3 \]
  \[ \vec{B}_0 + \vec{B}_1 \rightarrow \text{wiggles L/R} \]
- Circularly polarized field (quadrature)
  \[ \vec{B}_1^{\text{circ}} = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \]
  \[ = B_1 \cdot e^{-i\omega t} \]
  \[ \vec{B}_0 + \vec{B}_1 \rightarrow \text{rotates} \]
- In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF sin

Typical 90° flip (around x-axis)

Typical 180° flip (around opposite y-axis)
**SIGNAL EQUATION**

\[ \Phi(t) = \int_{\text{obj}} B(\hat{r}) \cdot M(\hat{r}, t) \, d\hat{r} \]

magnetic flux through coil (Scalar)

- For a particular instant in time vs. Block MxB which is change of M w/ t

- Ignoring the change in the z-component of the magnetization.
- Since it changes too slowly compared to the free precession of x- and y-components: \( \omega(r) \gg \gamma(\hat{r}) \)
- This is why we can only record transverse magnetization, \( M_{xy} \), but not longitudinal magnetization (\( M_z \) changes too slowly, so \( V(t) = \frac{d\Phi}{dt} \approx 0 \))

\[ V(t) = -\frac{d\Phi(t)}{dt} = -\oint_{\text{obj}} B(\hat{r}) \cdot M(\hat{r}, t) \, d\hat{r} \]

Faraday's law of induction

- Introduce local magnetization of object (time-dependent)
- Local field direction at each point onto coil magnetic field direction at each point
- Summed across object
- i.e., projection of magnetic vector at each point onto coil magnetic field direction at each point

\[ \Phi(t) = \oint_{\text{obj}} B(\hat{r}) \cdot M(\hat{r}, t) \, d\hat{r} \]

- Evaluate using numerical methods (solution to Bloch) ignoring relaxation
- Rewrite using complex notation in time-dependence from lab-frame Bloch

\[ S(t) = \int_{\text{obj}} M_{xy}(\hat{r}, 0) e^{-i \omega(\hat{r}) t} \, d\hat{r} \]

- Spatially-dependent resonant freq in rotating frame
- i.e., after subtraction of \( \omega_0 = \gamma B_0 \)

**Standard Signal Expression**

- Phase angle in rotating frame
- \( \omega t = \text{radians} \times \sec = \text{radians} \times (\phi / \omega t) \)
- Getting difference converts lab \( \rightarrow \) rotating frame

\[ w_{t} = \text{radians} \times \sec = \text{radians} \times (\phi / \omega t) \]
**PHASE-SENSITIVE DETECTION**

How we get rotating frame

\[ V(t) \xrightarrow{multiply} \text{PSD} \xrightarrow{\text{Low-Pass Filter}} S(t) \]

- method for moving very high frequency Larmor oscillations down to tractable frequency range

Demodulated signal \( \propto \) RF coil signal \( \cdot \) reference (transmitter)

\[
\propto \sin(w_0 + 2\pi ft) \cdot \sin w_0 t \\
\propto \frac{1}{2} \left[ \cos (2w_0 + \pi ft) - \cos (2w_0 + \pi ft) \right]
\]

This signal is digitized

One freq - freq domain

One freq - freq domain

Chirp - time domain

- two signals are made from a single receiving RF coil

- a quadrature coil can be treated the same way (OK to combine after adding \( \frac{\pi}{2} \) phase, then PSD)

- quadrature coil has better SNR since noise in each part is uncorrelated (\( \frac{1}{2} \) better)
**FID** — free induction decay, $T_2^*$  

$T_2$ — unrecoverable, rapid (intrinsic)  
$T_2^*$ — recoverable, static added

- **FID** (free induction decay) from an RF pulse w/angle $\alpha$
  
  $$S(t) = \sin \alpha \int_{w=-\infty}^{w=\infty} \rho(w) e^{-t/T_2^*(w)} e^{-i\omega t} \, dw$$

  - recorded signal (complex)
  - spectral density function
  - free, dependent, sum arias all freq
  - intrinsic decay
  - spin–spin

- For a single freq:
  
  $$S(t) = M_2^0 \sin \alpha e^{-t/T_2^*} e^{-i\omega t}$$

- Max FID amplitude at $t = 0^+$: $S(0) = M_2^0 \sin \alpha$

- If field inhomogeneous (Lorentzian distribution)
  
  $$S(t) = \pi M_2^0 \gamma \Delta B_0 \sin \alpha e^{-t/T_2^*} e^{-i\omega t}$$

  - bulk / transverse after flip
  - decay / oscillate
  - Lorentzian distribution

  Where
  
  $$\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$$

- Decay rate $1/T_2^*$ is increased

- Unrecoverable “intrinsic”

- Recoverable

- NB: actually complex

- Spectral density for “Lorentzian” mag. field inhomogen.
  
  $$\rho(w) = \frac{M_2^0 (\gamma \Delta B_0)^2}{(\gamma \Delta B_0)^2 + (w - w_0)^2}$$

- For other dist, assume $T_2^*$ is exp. approx.
  
  - See bottom

- Basic FID envelope is proportional to $e^{-t/T_2^*}$

- Both of these would be approximated by $T_2^*$

- Typical time course 10's of $\mu$s
c

  vs.

  each precession cycle, which is 10's of ms
**ECHOES — spin echo**

- Just after 90° $x'$ pulse $f_{x0} + f_{h}$ have same phase.
- Relaxation + phase dispersion of $f_{x0} + f_{h}$ (both from $B > B_0$).
- Just after 180° $y'$ pulse (y' pulse like x' pulse but RF has +90° phase).
- Echo caused by re-phasing of $f_{x0} + f_{h}$ (w/ decay due to $T_2$).

- Remember: RF just tips vector(s) while retaining length.
- Relaxation includes tips and shrinks (and grows for echo).
- 180° $x'$ pulse works, too, but echo will be $+\pi$ phase (left side in Figs above).
- Echoes generated even if second pulse not 180° (see next).

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**FID decay (and echo growth/decay) described by $T_2^*$ from inhomogeneity.**

- Reduction in height of echo compared to initial described by $T_2$.
- Echo fixes the star.
**Echoes** — Spin echo

- $\alpha_1 - \gamma - \alpha_2$ (both pulses along $y$, for simplicity)

**Effect of $\alpha$-pulse**

- $M_x' \rightarrow M_x' \cos \alpha - M_z' \sin \alpha$
- $M_y' \rightarrow M_y'$
- $M_z' \rightarrow M_x' \sin \alpha - M_z' \cos \alpha$

**Effect of $\tau$ delay**

- $M_x' \rightarrow (M_x' \cos \omega \tau + M_y' \sin \omega \tau) e^{-\gamma/2}$
- $M_y' \rightarrow (-M_x' \sin \omega \tau + M_y' \cos \omega \tau) e^{-\gamma/2}$
- $M_z' \rightarrow M_z' (1 - e^{-\gamma/2}) + M_z' e^{-\gamma/2}$

**Immediately after $\alpha$ pulse**

- $M_x'(w, 0^+): = -M_z'(w) \sin \alpha$
- $M_y'(w, 0^+): = 0$
- $M_z'(w, 0^+): = M_z'(w) \cos \alpha$

**After $\tau$ delay**

- $M_x'(w, \tau): = -M_z'(w) \sin \alpha \cos \omega \tau e^{-\gamma/2}$
- $M_y'(w, \tau): = M_z'(w) \sin \alpha \sin \omega \tau e^{-\gamma/2}$
- $M_z'(w, \tau): = M_z'(w) \left[1 - (1 - \cos \alpha \tau) e^{-\gamma/2}\right]$

**Immediately after $\alpha_2$ pulse** (no effect on $M_y'$; rewrite $y$; combine $x$ and $y$ eq's)

- $M_x'(w, \tau): = -M_z'(w) \sin \alpha \sin \omega \tau e^{-\gamma/2}$
- $M_y'(w, \tau): = M_z'(w) \left[1 - (1 - \cos \alpha \tau) e^{-\gamma/2}\right] \sin \alpha_2$

**Time dependent free precession around $z'$** (rewrite $M_x'(w, \tau)$)

- $M_x'y'(w, t): = M_x'y'(w, \tau) e^{-(\tau - t)/2} e^{-i\omega(t - \tau)}$

For a large num of freq's:

- $M_x'y'(w, t): = M_x'y'(w, \tau) e^{-t/2} e^{-i\omega(t - \tau)}$

  - $M_x'y'(w, \tau) = M_z'(w) \sin \alpha \sin \alpha_2 e^{-t/2} e^{-i\omega(t - \tau)}$
  - $M_z'(w) \left[1 - (1 - \cos \alpha \tau) e^{-\gamma/2}\right] \sin \alpha_2 e^{-(\tau - t)/2} e^{-i\omega(t - \tau)}$

**For FID of echo**

- Term 1: dephasing
- Term 2: nphasing

- $I(t) = \sin \alpha \sin ^2 \alpha_2 \int_0^\infty \rho(w) e^{-t/2} e^{-i\omega(t - \gamma/2)} dw$

**Echo Signal**

- $\rho(t) = \int_0^\infty \rho(w) e^{-t/2} e^{-i\omega(t - \gamma/2)} dw$

- $A_E = \sin \alpha \sin ^2 \alpha_2 \int_0^\infty \rho(w) e^{-\gamma/2} e^{-i\omega(t - \gamma/2)} dw$

**Peak Ampl.**

- $S_1(t) = \rho(t) e^{-t/2} e^{-i\omega(t - \gamma/2)} dw$
- $S_2(t) = \rho(t) e^{-t/2} e^{-i\omega(t - \gamma/2)} dw$

- $90^\circ - \gamma - 90^\circ$
- $S_1(t) = \frac{1}{2} \rho(t) e^{-t/2} e^{-i\omega(t - \gamma/2)} dw$

- $90^\circ - \gamma - 180^\circ$
- $S_2(t) = \rho(t) e^{-t/2} e^{-i\omega(t - \gamma/2)} dw$

- $90^\circ - \gamma - 180^\circ$
- $S_2(t) = \rho(t) e^{-t/2} e^{-i\omega(t - \gamma/2)} dw$

- Multiply by $i$ → add $\pi/2$ phase

**Echo Amplitude, ignoring freq dependence of $T_2$**

- $A_E = \frac{M_z'(w) \sin \alpha \sin ^2 \alpha_2 \frac{1}{e^{\pi/2}}}{}$
- Echo TRAINS - spin-echo trains

- It's (too) easy to make echoes ...

\[ E_n = \frac{3(\alpha - 1)}{2} \]

Echoes after end of nth pulse
3 RFs \rightarrow 4 echoes (here)
6 RFs \rightarrow 121 echoes (!)

Secondary echo: \[ SE_{1,2} \text{ acts like RF pulse} \]
\[ \alpha_2 \text{ makes an echo from it} \]

SE_{1,2}, SE_{2,3} - two more conventional two pulse spin echoes

Stimulated echo: combined effect of 3

\[ \alpha_1: M_L \rightarrow M_T \]
\[ \alpha_2: \text{leftover } M_T \text{ flipped to } M_L \text{ (saved)} \]
\[ \alpha_3: \text{flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays (after being held in limbo between 180° FID}_2 \text{ and FID}_3); \]
\[ \text{acts like 2-pulse echo} \]

- A useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2T spacing

\[ e^{-t/\tau_2} \]

- Typically, 90° and 180° applied in different axes (x', then y', y', ...)
which reduces phase errors due to imperfect 180° pulses
(since slightly off rotation around y' affects phase less)
EXTENDED PHASE GRAPHS

- Using full Bloch eq. solutions is tedious 😞
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize 90°, 180°)
- Problem #1: $\alpha$ pulse rotates a position of transverse magnetization into a position that results in rephasing
- Problem #2: third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

Rule for effect of $\alpha$
RF pulse on transverse mag

Rule for effect of $\alpha$
RF pulse on longitudinal mag

→ Echo when phase path crosses zero
HYPERS ECHOES

e.g., "SPACE"

1. 
- 3 solid lines
- 1 dashed line

2. 
- $\alpha_y - 180^\circ_y - \alpha_y \equiv 180^\circ_y$

3. 
- $\alpha_y \equiv 180^\circ_y$

Practical use

- multi-echo example

- practical prob: $180^\circ$ pulses deposit a lot of RF (6$\times$90°) $\Rightarrow$ prob at high fields

- by arranging to get big echo in middle of k-space can get by with much less RF power
**Gradient Echoes** — $T_2^*$, GE chains

- Initial negative gradient dephases spins
- After $t = T$ of positive gradient, spins rephase
- Does not correct for $T_2^*$ inhomogeneities
  - So echo amplitude is
    $$ A_E = e^{-t/T_2^*} $$
  - The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay
  - $\frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}}$ 
    $$ A_E = e^{-t/T_2^{**}} $$

- Key difference between spin-echo (SE) and gradient echo (GE) is that $B_0$ inhomogeneities not canceled
  - Hence, echoes are $T_2^*$-weighted, not $T_2$-weighted — more susceptible to inhomogeneities

- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get

- EPI hardware
  - $\Rightarrow$ 64 echoes
**IMAGE CONTRAST**

**T1 Saturation-recovery (no echo, just FID)**

- **Contrast** \((PD,T1,T2,T2^*)\) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

- RF

- \(M_z^o\): Longitudinal magnetization

- \(M_z\): Transverse magnetization

- **Steady state** after here

- **Simple Saturation/Recovery w/ no echo**

- **Initial conditions**:
  \[
  M_z^0 \text{ before first pulse} = M_z^0 \\
  M_z = 0 \text{ immediate after first pulse (i.e., 90° pulse)}
  \]

- From Bloch eq., \(M_z\) just before second pulse:

  \[
  M_z^{(2)}(t) = M_z^0 \left(1 - e^{-\frac{TR}{T1}}\right) + M_z^{(1)}(t) e^{-\frac{TR}{T1}}
  \]

- Given
  1. 90° pulse
  2. No \(M_{xy}\) left

- Pure tip: \(M_{xy} = M_z\)

- Tip existing mag

  \[
  M_z^0(t) = M_z^{(1)}(t) = M_z^0 \left(1 - e^{-\frac{TR}{T1}}\right)
  \]

- That is, the not-completely-recovered longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

- **Assume this is 0 because we assume that \(M_{xy}\) (transverse) completely decayed so that a 90° pulse doesn't generate any initial longitudinal.

- **I(r)**

  \[
  I(r) = C \rho(r) \left(1 - e^{-\frac{TR}{T1}(r)}\right)
  \]

  Spectral density \(\rho(r)\) at p. density; underlying equilibrium \(M_z^0\).
Why imperfect 90° takes multiple flips til steady state

- Initial fMRI images are usually discarded (why?)
  - Because they are brighter than all the rest
- Because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
  (e.g. at 3T, flip angle varies almost 25% across brain)

- At 3T, steady state
  - For typical 1-2 sec TR images reached after 8 images

\[ M_z^0 \]
**IMAGE CONTRAST**

IR (still just saturation-recovery — no echo)

- inversion recovery w/ no echo

RF

- 180° pulse reverses longitudinal magnetization
  \[ M_{z}^{(0)} = -M_{z}^{0} \]

- recovery to end of first TI from long. part of Bloch eq.
  \[ M_{z}^{'} = M_{z}^{0}(1 - 2e^{t / T1}) \rightarrow \text{flipped into transverse by second pulse (110 °)} \]

- longitudinal then regrows from zero (from first Bloch term only)
  \[ M_{z}^{'} = M_{z}^{0}(1 - e^{-(TR-TI)/T1}) \]

- after second 180°, just change sign again
  \[ M_{z}^{'} = -M_{z}^{0}(1 - e^{-(TR-TI)/T1}) \]

- apply relaxation eq. again
  \[ M_{z}^{'} = M_{z}^{0}(1 - e^{-TI/T1}) - M_{z}^{0}(1 - e^{-(TR-TI)/T1})e^{-TI/T1} \]

\[ M_{z}^{'} = M_{z}^{0}(1 - 2e^{-TI/T1} + e^{-TR/T1}) \]

\[ \text{this is magnetization flipped to transverse, made recordable} \]
- Steady state mag (2nd TR) just before 90°

\[ M_z'(0) = M_z^0 (1 - 2e^{- (TR-TE/2)/T1} + e^{-TR/T2}) \]

- the echo signal \( M_z \) unlike in simple saturation-recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation

\[ A_E = M_z^0 (1 - 2e^{-TR/2T1} + e^{-TR/T1}) e^{-TE/T2} \]

- if we assume TR much less than TR, then we can simplify:

\[ A_E = M_z^0 (1 - e^{-TR/T1}) e^{-TE/T2} \]

\[ \text{amplitude} \quad \text{echo} \]  
\[ \text{proton density} \quad \text{TR controls} \quad \text{T1 contrast} \]  
\[ \text{TE controls} \quad \text{T2 contrast} \]

- Similar equation for SE-IR

\[ A_E = M_z^0 (1 - 2e^{-TI/T1} + e^{-TR/T1}) e^{-TE/T2} \]
- Use basic longitudinal relaxation from Bloch eq. again

$$M_z^{(n)}(O) = M_z^0 \left(1 - e^{-TR/\tau_1}\right) + M_z^{(n-1)}(O) e^{-TR/\tau_1}$$

- Assume we have a small tip angle:

$$M_z^{(n)}(O) = M_z^0 \left(1 - e^{-TR/\tau_1}\right) + M_z^{(n-1)}(O) \cos \alpha e^{-TR/\tau_1}$$

- Assume we are in steady state:

$$M_z^{(n)}(O) = M_z^{(n-1)}(O) = M_z^{ss}(O)$$

**Pre-pulse**

$$M_x^{ss}(O) = \frac{M_z^0}{1 - \cos \alpha e^{-TR/\tau_1}}$$

**Post-pulse**

$$M_x^{ss}(t) = \frac{M_z^0 \left(1 - e^{-TR/\tau_1}\right)}{1 - \cos \alpha e^{-TR/\tau_1}} \cdot \sin \alpha e^{-TE/\tau_2}$$

**Gradient echo amplitude**

$$A_E = \frac{M_z^0 \left(1 - e^{-TR/\tau_1}\right) \sin \alpha e^{-TE/\tau_2}}{1 - \cos \alpha e^{-TR/\tau_1}}$$

TI contrast mostly depends on flip angle, not TR $\rightarrow \cos \alpha \approx 1$ $\rightarrow$ eliminates TI weight since denon $=$ numer.
- Saturate, wait for contrast $T_1$, invert, wait for contrast $T_2$, Flash (continued)

A) $M_x' (\text{just after } 90^\circ) = 0$ (perfect $90^\circ$)

B) $M_x' (\text{after TD}) = M_x^0 (1 - e^{-TD/T_2})$ (Bloch term #1)

C) $M_x' (\text{just after invert}) = \cos \phi M_x^0 (1 - e^{-TD/T_2})$

D) $M_x' (\text{after TI}) = M_x^0 (1 - e^{-TI/T_1}) + [\cos \phi M_x^0 (1 - e^{-TD/T_2})] e^{-TI/T_1}$

After preparation:

Special Case $T_1 = T_2$: $= M_x^0 [1 - (1 - \cos \phi (1 - e^{-TD/T_2}) e^{-TI/T_1})]$

- After the first $\alpha$ pulse:

E) $M_x' (\text{just after pulse}) = M_x^0 [1 - (1 - \cos \phi (1 - e^{-TD/T_2}) e^{-TI/T_1})] \sin \alpha$
QUANTITATIVE T1 - HELMS Z-FLIP ANGLE METHOD

- start w/ gradient echo signal e.g., dropping T2 decay: \( e^{-TE/T2} \)
  \[
  S_{E_{\text{est}}} = A \cdot \sin \alpha \cdot \left( 1 - e^{-TR/T1} \right) \frac{1 - \cos \alpha \cdot e^{-TR/T1}}{1 - \cos \alpha \cdot e^{-TR/T1}}
  \]
  Ernst eq.
  (max: \( \cos \alpha_{\text{E}} = e^{-TR/T1} \))
  \( \sim \) "Ernst angle"

- simplify/linearize/estimate
  \( TR \ll T1 \)
  linear approx. of exponentials
  Taylor expansion simplification of \( \sin, \cos, \) drop small term

  Helms et al. (2008)
  \[
  S \approx A \alpha \frac{TR/T1}{x^2/2 + TR/T1}
  \]
  (max: \( x^2/2 = TR/T1 \))

- solve for TD and
  \( A \) (proton-density) given
  signals from 2 different flip angles

  \[
  T1_{\text{est}} = 2TR \frac{S_1/x_1 - S_2/x_2}{S_2/x_2 - S_1/x_1}
  \]

  \[
  A_{\text{est}} = S_1S_2 \frac{(x_2/x_1 - x_1/x_2)}{(x_2/x_1 - x_1/x_2)}
  \]

- problem: flip angle varies a lot
  at 3T (e.g., 25%) from nominal/requested (e.g., flip series)

- collect spin-echo and stimulated echo (FPI)
  estimate T1/T2
  \( \text{solve for } 2 \alpha \) insert
  \( \begin{align*}
  S_1 &= k_1 \sin^3 \alpha \cdot e^{-TE/T2} \\
  S_{STM} &= \frac{1}{2} \sin^3 \alpha \cdot \sin 2 \alpha \cdot e^{-TE/T2 - TM/T1} \\
  \alpha &= \arccos \left( \frac{S_{SE} \cdot e^{-TM/T1}}{S_{STM}} \right)
  \end{align*} \)

  Jiru & Klose (2006)
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: 
  \[ SNR = \frac{\text{avg object signal}}{\text{s.d. non-object region}} \]

- Temporal SNR: 
  \[ S_{\text{SNR}} = \frac{S}{\sigma} \]

- "Contrast" is a difference

- Contrast-to-noise ratio: 
  \[ \text{CNR}_{AB} = \frac{S_A - S_B}{\sigma} = \text{SNR}_A - \text{SNR}_B \]

- Spin-echo: 
  \[ A_E = M_e^0 (1 - e^{-TR/T1}) e^{-TE/T2} \]

- Gradient echo: 
  \[ A_E = \frac{M_e^0 (1 - e^{-TR/T1}) \sin \alpha}{1 - \cos \alpha \ e^{-TR/T1}} e^{-TE/T2\alpha} \]

**Lactaburt. tissue**

<table>
<thead>
<tr>
<th>T1</th>
<th>T2</th>
<th>PD</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>450</td>
<td>2200</td>
</tr>
<tr>
<td>WM</td>
<td>600</td>
<td>80</td>
</tr>
<tr>
<td>CSF</td>
<td>1200</td>
<td>100-200</td>
</tr>
</tbody>
</table>

**General rules:** spin-echo, long TR GE

<table>
<thead>
<tr>
<th>Proton-density weighted</th>
<th>TR high (no T1 diffs)</th>
<th>TE high (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>TR = Ti (big T1 diffs)</td>
<td>TE = T2 (no T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>TR = Ti (no T1 diffs)</td>
<td>TE = T2 (big T2 diffs)</td>
</tr>
</tbody>
</table>
SIGNAL-TO-NOISE $S/N$

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \Delta x \Delta y \Delta z \sqrt{\frac{N_z N_x N_y \Delta N}{N_z \Delta N}}
\]

- Size (volume) of voxels (with the number of voxels held constant), linear effect on $S/N$

\[
\text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better } S/N
\]

- More voxels (with size of voxels, $\Delta t$ per read step constant), $\sqrt{n}$ effect on $S/N$

\[
\text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{\sqrt{128 \times 128}}{\sqrt{64 \times 64}} = 2 \text{ times better } S/N
\]

- # acquisitions $\sqrt{n}$ better $S/N$

\[
\text{e.g., } 1 \text{ acq } \rightarrow 2 \text{ acq } \rightarrow \frac{\sqrt{2}}{1} = 1.41 \text{ times better } S/N
\]

- Larger timestep during readout, $\sqrt{\Delta t}$ better $S/N$

\[
\Delta t = \frac{1}{\text{BW}_{\text{read}}}, \text{ digitization timestep during echo acquisition}
\]

- BW$_{\text{read}}$ determined by cutoff freq, analog low-pass filter
- $\Delta t$ controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) $\Delta t$
- Must filter out freq's $> f_{\text{max}} = \frac{1}{2\Delta t}$ because they alias
**COMPLEX ALGEBRA**

- **Real/Imaginary**
  - add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
  - mult: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

- **Angle/Phase**
  - add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
  - mult: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2, \phi_1 + \phi_2)\) → N.B.: 3rd kind of vector mult., different than dot product and cross product
  - complex to real power: \((A, \phi)^n = (A^n, n\phi)\)

**Real to Complex Power**

\[ e^{i\phi} = \begin{cases} 
\text{expand as series} \\
\text{recognize cos, sin series} \\
= \cos \phi + i \sin \phi \\
= \cos \phi, \sin \phi \\
= \text{vector on unit circle} \\
\end{cases} \]

\[ e^{i\phi^n} = (\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi \]

**Fourier Transform**

\[
H(t) = \int \limits_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt
\]

\[
H(\omega) = \int \limits_{-\infty}^{\infty} h(t) e^{j2\pi \omega t} dt
\]

**Convolution Theorem**

\[
F[g(x) \ast h(x)] = G(k) \ast H(k)
\]

because of FFT, faster if kernel not small

**Convolution**

\[
f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) dz
\]

**Cross-correlation**

\[
f(x) = g(x) \otimes h(x) = \int g(z) \cdot h(x+z) dz
\]

- For arbitrary amplitude, multiply \(A \cdot e^{i\phi}\)
- Phase is integral of freq. variable \(\phi = \int \omega \, dt\)

- Shorthand for a unit vector \(e^{i\phi}\) pointing in the direction of \(\phi\)
Fourier transform (1) for one $t$:

- How to calculate $H(t)$ for one $f$ ($t=3$):

Real signal

Imaginary signal

$cos$ $e^{-i \omega t}$

$sin$

$i.e., \text{two correlations}$

Real frequency domain

Imaginary frequency domain

$3$ $f$ $6$

$6$

Cartesian $(A, \phi)$

Polar co-ords $(A, \phi)$

Amplitude frequency domain

Phase frequency domain

Like correlating with $\sin$ and $\cos$ (at each freq) so we get phase (at each freq.)
Fourier transform (1b)

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} \]
\[ = \cos (-\phi) + i \sin (-\phi) \]
\[ = \cos \phi - i \sin \phi \]

- \(\cos\) is an "even" function, \(\sin\) is an "odd" function

\[ \begin{align*}
\text{even} & \quad \sin(x) \\
\downarrow \text{equals} & \quad \sin(-x) \\
\cos(x) & \quad \downarrow \text{flips} \\
\cos(-x) & \quad -\cos(x)
\end{align*} \]

\[ \begin{align*}
\text{odd} & \quad \cos(x) \\
\downarrow \text{flips} & \quad \cos(-x) \\
\sin(x) & \quad \downarrow \text{equals} \\
\sin(-x) & \quad -\sin(x)
\end{align*} \]

An orthogonal decomposition

- think of discretely sampled \(\sin(bx), \cos(bx)\) as vectors
- \(\text{Corr}(\vec{v}_1, \vec{v}_2) \equiv \) projection of \(\vec{v}_1\) onto \(\vec{v}_2 \equiv \vec{v}_1 \cdot \vec{v}_2\)

\[ \begin{align*}
\text{Corr} (\cos bx, \sin bx) &= 0 \\
&= \sin \text{ and } \cos \text{ of same frequency are orthogonal} \\
&= \sin 2x \quad \cos 2x
\end{align*} \]

\[ \begin{align*}
\text{Corr} (\sin bx, \sin bx) &= 0 \\
&= \text{different integer freqs of } \sin \text{ and } \cos \\
&= \sin 2x \quad \sin 3x
\end{align*} \]

\[ \text{Corr} (\cos bx, \sin bx) = 0 \quad \text{[as above]} \]

- in the continuous case, orthogonal functions defined as:

\[ \int_{x=hi}^{x=lo} f(x) g(x) \, dx = 0 \]
Understanding Inverse Fourier Transform as Another Case of Corr w/ Cos, Sin

- Start with spike in image domain
- Take example of spike at $x = 0$
  \[
  \left[ \cos(x), \cos(2x), \cos(kx) \right] \text{ all equal 1 there.}
  \left[ \text{all freqs correlate w/ spike at } x = 0 \right]
  \]
- If spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates
- To see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of $e^{-2\pi i k x}$ cos and sin at location of spike
- Cos (even), Sin (odd)
  \[
  \left[ \text{positive spikes same dist from origin: } \Rightarrow \text{ pick cos's}
  \right.
  \left[ \text{positive & negative spikes, same dist: } \Rightarrow \text{ pick sin's} \right]
  \]
- This is one way of thinking about what one point in k-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

(1) real image  imaginary image  (Zero)

(2) amplitude image  phase image  (Zero)

(3) complex vectors

-3 equivalent representations of image & spat. freq. space
FOURIER TRANSFORM OF AN IMAGE (3)

- What a single k-space point looks like in image space (polar coordinates A, φ instead of r, i)

image space

k-space (spatial freq. space)

Offset of stripes is k-space phase

Brightness of stripes is k-space amplitude

Distance from center is stripe spacing

Angle of point is angle of stripes

Amplitude

(rest all 0)

Phase

(rest all 0)

Inverse Fourier transform

(image recon.)

N.B. each dimension of k-space — x- and y- spatial freq.

N.B. increasing one 1D component increases the spatial freq of the 2D wave and rotates it

Cartesian dimension of k-space — x- and y- spatial freq

(Should be all zero not same as "stripe phase" above)

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin, cos — don't confuse k_x, k_y w/ sin, cos!
Fourier Transform of Image (4)

- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)
- Example: Cosinusoid in image space, then shifted in x-dir

**REAL IMAGE**

$I(x,y) = \cos(x)$

**FT of REAL IMAGE**

$I(x,y) = \cos(x - \frac{\pi}{4})$ → half way between cos and sin (Shifted 45° to right)

$\text{real component less than above because rot}$
FOURIER TRANSFORM OF IMAGE (S)

- (cont.) center of k-space (real image)
- complex image

**REAL IMAGE**

\[ I(x, y) = 1 + \cos(x) \]

\[ \begin{bmatrix}
  z & 0 \\
  0 & z
\end{bmatrix} \]

\[ \begin{bmatrix}
  1 & \cos(x) \\
  \sin(x) & 0
\end{bmatrix} \]

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**Gradient Coils**

- Gradient coils for x, y, z generate approximately a linear gradient in the strength of the z-component of the magnetic field \( B_2 \).

- For example, the x gradient coil induces a ramp in the z-component of the magnetic field when moving in the x-direction:

\[
B_{g,z} = G_x x
\]

* Since a pure linear gradient of \( B_{g,z} \) in only the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-direction (\( B_{g,x} \) and \( B_{g,y} \)).

- The other magnetic field components are usually ignored because they are so small relative to \( B_{g,z} \), since \( B_{g,z} \) is added to \( B_0 \), and since \( B_0 \) is much stronger than \( B_{g,x} \), \( B_{g,y} \), and \( B_{g,z} \).

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- The Maxwellian terms \( B_{g,x} \) and \( B_{g,y} \) are known; can be included in the reconstruction process.
SLICE SELECTION ($G_z$)

- slice select gradient on during RF stim

\[ f = \frac{G_z}{\gamma} \left( B_0 + B_{G_z} \right) \]

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq. encode) these have to be removed by a post-excitation rephasing $z$-gradient

- approximation from assuming tip occurs instantaneously in middle

- valid for small tip: $90^\circ \rightarrow 52%$

- in practice: adjust to max, use crusher to kill spurious transverse on $180^\circ$
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx. even tho tipping is non-linear)

\[ B_1(t) \propto \int_{-\infty}^{\infty} p(f) e^{-i2\pi ft} df \]

- Time dependent
- RF stimulation (complex)

Solve with: \( p(f) = \text{frequency band} \)

\[ B_1(t) = A \cdot f_w \cdot \text{sinc} \left( \frac{\pi f_w t}{\text{FWHM}} \right) e^{-i2\pi ft} \]

- Amplitude controlling flip angle
- Frequency determined by sinc width
- Sinc envelope width inversely proportional to \( f_w \)
- Modulation (complex) at center freq. \( f_c \)

Fourier Transform Pair, Rules:
- Convolution in one domain is multiplication in the other
- Convolution with delta function, impulse moves function to impulse center

Fourier Transform Solution to: \( \frac{d}{dt} \)

\[
\begin{align*}
\text{Frequency} & \leftrightarrow \text{Time} \\
\ast \text{convolution} & \leftrightarrow \times \text{multiply} \\
\text{equals} & \leftrightarrow \text{equals}
\end{align*}
\]
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

**Spectroscopy**

1) chemical shift change freq. → \(\text{gradient changes freq.}\)
2) stimulate w/ broadband RF → same
3) time-sampled FID containing multiple frqs. → same
4) FT of FID to get spectrum \(\frac{\Delta f}{\Delta t}\) of \(\Delta f\) offsets \(\rightarrow\) FT of FID to get \(\Delta x\) offsets

- this is technically correct (FT of FID) but highly misleading
  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect frqs in FID

- the "k-space" perspective is a "Copernican Turn"
  - idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift-like frequencies
  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations (which are analogous to multiple time points)

- i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>spectroscopy</td>
<td>samples of oscillations in time-domain FT (\rightarrow) frequency-domain spectrum of shifts</td>
</tr>
<tr>
<td>MRI</td>
<td>samples of spatial freq. in freq-domain FT(-1) (\rightarrow) spatial object (like a time-domain signal)</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because \(\text{FT} \cong \text{FT}\(^{-1}\) (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient \((G_x)\) causes precession rates to vary linearly in \(x\)-direction

\[
\begin{array}{c}
\uparrow \text{precession} \\
\uparrow B_z \\
\text{in} \\
x \text{-direction} \\
x \rightarrow
\end{array}
\rightarrow \text{correct} \quad \text{(remember that strength of } G_x \text{ causes variation of slope of } B_z \text{ in } x \text{-direction)}

- Different frequency signals are mixed together and recorded as a 1-D signal over time

\rightarrow \text{correct, but remember, we are recording summed local magnetization vectors after de-modulation}

- A Fourier transform, which converts back and forth between \(x\)-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal

\rightarrow \text{correct}

- Spatial frequencies get confused/confused with precession frequencies

\rightarrow \text{wrong !!}

Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

\rightarrow \text{conceptually wrong !!}

\rightarrow \text{\(\mathcal{F}\) actually converts spatial frequencies to spatial position}

\rightarrow \text{the spatial frequency increases for each time point in the readout}

\rightarrow \text{the precession freq ramp is constant each timestep}
FREQUENCY ENCODING (2) connect intuition - why phase critical

- "Frequency"-encode gradient \( G_x \) turned on during
  during echo causes precession rates
  to immediately vary with \( x \)-position

\[
G_x \quad \uparrow B_z \text{ in } x\text{-direction}
\]

- at beginning of gradient on, the phase of
  signal coming from each \( x \)-position is the same
  \text{summed phase angle is what we measure}

- early after gradient on, phase advances (because
  of faster precession frequency) arise with greatest
  phase advance at largest \( x \)-position

\[
\text{one cycle of spatial frequency of phase angle (= low spatial freq)}
\]

- later during gradient on, phase advances cause
  multiple wrap-arounds of phase angle across space

\[
\text{multiple cycles of spatial frequency of phase angle (= hi spatial freq)}
\]

- in practice, the lowest spatial frequency (= 0)
  occurs in the middle of the gradient on time
  because the phase is "gained" negatively by
  a preparatory gradient (to the highest negative
  spatial frequency) before data collection occurs

\[
\theta = \text{max negative} \quad \theta = 0 \quad \theta = \text{max positive}
\]

\( G_x \) is spatial frequency

\( t \) is timepoint

\( \theta \) is phase angle

\( x \) is spatial position
FREQUENCY ENCODING (3) why each datapoint is 1 spatial freq

Standard Fourier transform: (Temporal freq ↔ time)

\[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-i 2\pi ft} dt \]

"k" is often used instead of "f" for the frequency variable

 Imaging equation: (Spatial freq ↔ space)

\[ S(f) = \int_{-\infty}^{\infty} I(x) e^{-i 2\pi fx} dx \]

Sum across x of object

done by RF coil recording sum

Signal strength at one x-position (brightness of image point)

Oscillations come from readout phase wrapping, where f is single spatial freq (e.g., 5) and x goes across object

End: \( f = G_x \), that is, spatial freq depends on amount of time gradient was on (this \( f \) increases with time)

don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each x position)

To make image, do inverse Fourier transform of recorded signal, \( S(f) \)

one data point (i.e., one spatial freq) during readout (2 components)

G_x

RF

get this single readout point by summing signal across x-position (RF coil records sum)

even though variable is \( f \), it represents one time period during readout
**ALTERNATE DERIVATION (incl. effects of \( G_x \)) SIGNAL EQ**

- oscillators at \( \omega = \gamma B \) at each position (just \( x \) for now)

\[
S(t) = m(x) e^{-i \phi(x)} dx
\]

- by definition, freq, \( \omega \) is rate of change of phase, \( \phi \)

\[
\frac{d\phi(x,t)}{dt} = \omega(x,t) = \gamma B(x,t) \quad \text{and} \quad \phi(x,t) = \int_0^t \omega(x,t) dt = \gamma \int_0^t B(x,t) dt
\]

- assuming phase initially 0, \( B \) affected by gradients

\[
B(x,t) = B_0 + G(x,t) \cdot x
\]

\[
\phi(x,t) = \gamma \int_0^t B_0 dt + \left[ \gamma \int_0^t G_x(t) dt \right] x
\]

\[
= \omega_0 t + 2\pi k_x(t) x
\]

\( k \) is time integral of gradient waveform

- demodulation removes the \( B_0 \)-caused carrier frequency \( e^{-i\omega_0 t} \) from the first equation

\[
S(t) = \int_x m(x) e^{-i \omega_0 t - i 2\pi k_x(t) x} dx
\]

amplitude of each oscillator

gradient-controlled phase
Phase-Encode Gradient $G_y$

- Turn on gradient after excitation but before readout.
- Different levels of $G_y$.
- Higher levels of $G_y$ (slope of $B_z$ in y-direction!)
- Higher spatial freq. (more phase wraps) in y-direction.
- Phase wraps persist after phase-encode gradient off.
- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase.

2D Imaging Equation

$$S(k_x,k_y) = \int \int \int \int I(x,y) \cdot e^{-i2\pi(k_x x + k_y y)} \, dx \, dy$$

- Signal recorded at single time point (one readout point).
- Complex signal (from phase-sensitive detection).
- Done by RF coil.
- Sum across x-y plane.
- Scalar (what we try to reconstruct).
- Phase angle (of scalar magnetization!) in rotating frame, set by gradients.

- Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia"—they stay wherever the gradients last left them.
3-D IMAGING — two phase-encode gradients

- use z-gradient for 2nd phase-encoding instead of slice selection
- excitation of whole slab (slice-select is whole brain)
- simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

Simple 3D spin echo example

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)
- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice
- phase stripes created throughout volume vs. slice:

\[ S(k_x, k_y, k_z) = \int \int \int \delta(x-x_0, y-y_0, z-z_0) e^{i2\pi(k_x x + k_y y + k_z z)} \, dx \, dy \, dz \]

\[ I(x,y,z) = e^{-i2\pi(k_x x + k_y y + k_z z)} \, dx \, dy \, dz \]

N.B., this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

- Since the phase-encode gradient and the freq. encode gradient both affect phase the result is a rotation of phase "stripes" when the two add.

- Successive read out steps:

  - Small phase encode $G_y$
    - 3D phase encode w/ $G_y$ and $G_z$ starts rotated in y-z plane
  - Large phase encode $G_y$

- More rotation higher spatial freq

  - Phase of whole image summed to one (complex number) by RF coils

  - N.B.: Stripes have sharp edges from phase wrap (not sinusoidal sine of from 2-comp quadrature!)

  - Stripes here represent complex value

  - Phase of whole image summed to one (complex number) by RF coils

  - E.g., after x-gradient, spins at a point might be 3 cly ahead while after y-gradient spins at same point 7 cly ahead: but counting wraps in x-direction, still only 3 ahead
GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point.

\[ k = \frac{\sigma}{t} \int G(t) \, dt \]

Spatial frequency recorded at \( t \) = record time function of \( t \)

Simple form up integral w/ boxcar gradient

record data point here

- all of the following gradients end up at the same point in k-space:

Frequency-encode

**FID**

RF \( 90^\circ \)

\( G_x \)

samples

\[ k_y \quad k_x \]

Frequency-encode gradient echo

RF \( 90^\circ \)

\( G_x \)

\[ k_y \quad k_x \]

Frequency-encode spin-echo (plus gradient echo!!)

RF \( 90^\circ \) \( \tau \) \( 180^\circ \) \( \tau \)

\( G_x \)

RF \( 180^\circ \)

\( G_x \)

Neg \( G_x \)

Pos \( G_x \)

Nyq.: 180° wave to conjugate point

Phase-encode then frequency encode gradient echo

RF \( 90^\circ \)

\( G_y \)

\( G_x \)

\( G_x + G_y \)

\[ k_y \quad k_x \]
**IMAGE RECONSTRUCTION**

\[ S(k_x, k_y) = \int \int \int \int I(x, y) e^{i 2\pi (k_x x + k_y y)} dx dy \]

\[ I(x, y) = \int \int S(k_x, k_y) e^{i 2\pi (k_x x + k_y y)} dk_x dk_y \]

In practice, the image is real in the ideal case, but complex in practice. Use the amplitude image.

Adding exponents is the same as multiplying two \( e^{i 2\pi k x} \)'s.

In practice, a finite number of samples, \( N \) and \( M \), are collected in the \( k_x \) and \( k_y \) directions of \( k \)-space (integral \( \rightarrow \) discrete sum).

\[ I(x, y) = \sum_{|x| < \frac{1}{\Delta k_x}} \sum_{|y| < \frac{1}{\Delta k_y}} S(n, m) e^{i 2\pi n \Delta k_x} e^{i 2\pi m \Delta k_y} \Delta k_x \Delta k_y \]
**Sampling**

- must consider effects of sampling
  - limited points in k-space
  - limited in range of frequencies sampled (kmin → kmax)
  - limited in rate of sampling (Δk)

- N.B., aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

- **Correct reconstruction**

- **Correct plus replicas**

- **as above w/ blurring, ringing**

- **as above w/ blurring, ringing**

- aliasing occurs in spatial domain
  - replicas overlap, causing wraparound

- **Thus, finer sampling of same range of spatial freqs increases FOV**
**UNDER/OVER SAMPLE**

More examples

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]

\[ \delta_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x} \]

FOV (distance to repeat) is reciprocal of spatial frequency sampling interval

Pixel size is FOV divided by K-space sample count

### 3 More Examples (not incl. less samples to same spat. freq. on bottom last page)

- **Basic Image**
- **Same num samp. to 2X spat. freq.**
  - (i.e. gradients stronger or time ON longer)
- **2X num. samples to same spat. freq.**
  - (i.e. gradients weaker or time ON shorter)
- **2X number samples to 2X spat. freq.**
  - (i.e. gradients stronger or time ON longer)

**Spatial Freq: K-space**

- \( N = 10 \)
  - \( k_x = 5 \)
  - \( \Delta k_x = 1 \)
  - \( \text{FOV}_x = 0.1 \)

- \( N = 10 \)
  - \( k_x = 10 \)
  - \( \Delta k_x = 2 \)
  - \( \text{FOV}_x = 0.05 \)

- \( N = 20 \)
  - \( k_x = 5 \)
  - \( \Delta k_x = 0.5 \)
  - \( \text{FOV}_x = 0.1 \)

- \( N = 25 \)
  - \( k_x = 10 \)
  - \( \Delta k_x = 1 \)
  - \( \text{FOV}_x = 0.05 \)

***Space***

- Basic image
- Square pix
- \( -x\)-pix half width
- Replicas intrude

Scanner makes square image

"Wrap" occurs

- Square pix
- Twice \( x\)-pix count
  - so \( \text{FOV} = 2X \)
- This is "phase oversamp"

Scanner crops to square

Replicas move out

- \( -x\)-pix half width
- Twice \( x\)-pix count
- Same \( \text{FOV} \)
- This is decrease

Pixel size w/o

Change FOV
Fourier Transform Solution to Replicas

1. Sample data freq
   spatial freq
   \[ \ldots \uparrow \uparrow \uparrow \uparrow \uparrow \ldots \]
   \[ \leftrightarrow \]
   \[ \xrightarrow{\text{multiply}} \]
   \[ \ldots \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \ldots \]

2. Result in image space
   space
   \[ \ldots \uparrow \uparrow \uparrow \uparrow \uparrow \ldots \]
   \[ \leftrightarrow \]
   \[ \xrightarrow{\text{convolution}} \]
   \[ \ldots \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \ldots \]

3. equals

- Limit approach to Fourier transform of conv

Fov = \( \frac{1}{\Delta k} \)

\( \Delta k = \frac{1}{\text{Fov}} \)

Useful FTs

- Rect
  \[ \text{Rect} \left( \frac{x}{W} \right) \xrightarrow{\mathcal{F}} W \text{sinc}(\pi Wk) \]

- Gaussian
  \[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \]

- Comb
  \[ \sum_{n=-\infty}^{\infty} \delta(x - n\Delta k) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) \]
**POINT-SPREAD FUNCTION**

\[ \hat{I}(x) = \Delta k \sum_{n \in \mathbb{N}^2, \mathbb{N}^3} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:
  \[ S(m \Delta k) = 1 \]
  
- Substitute into \( \hat{I} \) to get PSF:
  \[ h(x) = \Delta k \sum_{n \in \mathbb{Z}^2, \mathbb{Z}^3} e^{i 2\pi n \Delta k x} \]
  
- Simplify:
  \[ h(x) = \Delta k \frac{\sin(\pi N \Delta k x)}{\sin(\pi \Delta k x)} \] \( \Rightarrow \) periodic

- That is, image is reconstructed from a sum of sincs, because the FT of a boxcar pixel in \( k \)-space is an image sinc.

---

**Image**

- How PSF modifies ideal (infinite \( k \))
- Image

\[ \hat{I}(x) \]

- Convolve

**Frequency**

- FT \( \times \) multiply

- Acquisition window (truncated in spatial \( b \))
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \int_x I(x) e^{-i \frac{2\pi}{\lambda} k_x x} dx \]

Signal eq. \( \rightarrow \) fwd problem

\[ I(x) = \int_{k_x} S(k_x) e^{i \frac{2\pi}{\lambda} k_x x} dk_x \]

Recon eq. \( \rightarrow \) inv. problem

\[ s = F^* \]

Linear "forward solution"
- Matrix/vector have complex entries
- Can build in any measurable/priors

\[ F_{x,y,z} = g(x,y) \quad e^{-\frac{(nT \pm m\Delta T + TE)}{T_2}} \quad e^{-i \gamma B_0 (nT \pm m\Delta T)} \quad e^{-i \frac{\gamma B_0 (n\Delta k_x + m\Delta k_y)}{\Delta k_z}} \]
- T2 decay
- B0 error
- T2 + phase

Multi-coil

\[ s \]

\[ \begin{bmatrix}
    s_1 \\
    s_2 \\
    s_3 \\
    \vdots \\
    s_N \\
\end{bmatrix} = \begin{bmatrix}
    F_{\text{coil1}} \\
    F_{\text{coil2}} \\
    \vdots \\
    F_{\text{coilN}} \\
\end{bmatrix} \begin{bmatrix}
    i_1 \\
    i_2 \\
    \vdots \\
    i_N \\
\end{bmatrix} \]

- Naturally incorporates undistorted field map
- Different sensitivity function for each coil
- Contains additional info about source loc., but, need reference scan, low-res ok
- Need phase corrections for each coil

\[ i = F^* s \quad \text{over-determined} \]

More precise inverse:

\[ F^* = \left( F^T F \right)^{-1} F^T \]

\[ (x,y)^2 \rightarrow \text{"small"} \]

\[ (x,y,\text{ coil})^2 \rightarrow 16 \times \text{ bigger for 4 coils} \]

\[ i = \left( \left( F^T F \right)^{-1} F^T \right) s \]

Slice-by-slice
- Assume slice select sweeps negligible
**FAST SPIN ECHO (FSE)**

- One 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient.

- Each phase "winder" is "unwound" because leftover phase would be re-focused away by 180° (vs. FSE where it persists between blips).

- The effective TE is the TE when center of k-space is collected (largest effect on contrast, largest echo).

- Each subsequent echo has more T2 decay: \( E_n = e^{-n\text{TE}/T2} \) for \( n = 1, 2, \ldots, M \).

- By arranging to collect \( ky = 0 \) early, PD-weighted instead of T2-weighted.

- Possible to correct different T2-weighting of echoes by estimating T2 curve from \( Gy = 0 \) echo train.

- 3D FSE — like 2D except wind/unwind added to thick slice select (w/ emskers on 180°).
MULTI-SLAB 3DFSE, PROBLEMS

- echo train e.g. 20
- etc to fill 3D k-space

- $G_z$ is "partition"
- $G_y$ is "phase encode"
- $G_x$ readout needs no pre-wind since $180^\circ$ does it
- $T_{\text{eff}}$ is $90^\circ$ to echo thru center of k-space
- echoes die out quickly $\propto e^{-t/\tau}$
- since echoes after $90^\circ$ limited to $<30$, can't fill 3-D k-space in a reasonable time
- SAR constraint $\text{SAR} \propto B_0^2 \theta^2 \Delta \phi$
  $\Rightarrow 180^\circ$ pulses deposit 4-6x power of $90^\circ$

- "multi-slab" is halfway between slices and single-slab

- problem at slice boundaries — esp. movement
- multislab requires slice selective RF pulses $\rightarrow$ longer than non-selective 'hard' pulses

- 4 ms RO
- limits speed of covering k-space

- hard to get under 8 msec inter-echo spacing
SINGLE-SLAB 3D FSE

- regular FSE (180° pulse train)
- sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
- this "storage" in Z-axis preserves magnetization for longer time
- smaller flip angles allow much longer echo trains
- enough to collect whole plane of 3-D k-space
- different than hyper echoes (not symmetric)
- contrast must consider STE

\[ SE = \sin \alpha, \sin^2 \frac{\alpha^2}{2} e^{-2\pi/12} \]
\[ STE = \frac{1}{2} \sin \alpha \sin^2 \alpha e^{-T/12} e^{-2\pi/12} \]

SE = spin echo plus stimulated echo

FID spin echo 15° FID A spin echo plus stimulated echo

NB: time to collect k-space is \( k \times 5X \)
apparent contrast time b/c of "storage"
(e.g. \( TE_{eff} = 585 \, ms \) looks like FSE \( TE = 140 \, ms \))
FAST GRADIENT ECHO (GRASS | FLASH | FISP | SPGR | MPRAGE)

- Small tip so TR can be greatly reduced (e.g. 10 msec, less than T2)
- ‘Leftover’ un-decayed transverse magnetization "unwound" and re-used "spoiled" before next shot

STEADY-STATE COHERENT (GRASS, FISP)

- Unwind phase from phase-encode M_r before next pulse (there because TR < TE)
- Unwind read gradient, too

\[ S = k \sin \left( \frac{1}{1 + \cos \alpha + (1 - \cos \alpha) \frac{T_r}{T_2}} \right) e^{-\frac{T_r}{T_2}} \]

\[ T_2/T_1 - \text{weighted contrast (bright CSRF)} \]

STEADY-STATE SPOILED (SPGR, FLASH)

- Spoil with random gradient (but this still allows some α refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T1-weighted)

NON-STEADY-STATE, MAGNETIZATION-PREP

(MPRAGE)

- Longitudinal mag. not affect much by low angle pulses

- Preparation pulse → Strong T1-weighting
- Contrast varies in spatial - freq. dependent way

[ prep. pulse on spin echo sequence (k_y = 0) ]

Effective TI actually time to TR that records signal

Nominal inversion time

TR ∼ 10 msec

TR ∗ 10 msec
ECHO PLANAR IMAGING, EPI

- Single shot EPI collects all k-space lines (e.g. 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space).

- Therefore, the recording point (∆t) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the Gy "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of K-space, which are only recorded after about 32 echoes.
**SPIN ECHO EPI**

- Why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing

- The excess of oxyhemoglobin (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect

- Spin echo cancels static $T_2^*$ ($T_1'$) dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For TE $\approx 100$ ms, spins diffuse 10's of $\mu$m, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion is less likely to expose spin to different fields here)

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intr/extra v/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels

- Over half of SE-BOLD at 1ST is venous...
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence.
- "Spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space.
- "asymmetric spin-echo EPI" arranges for the spin echo to occur T msec before the gradient echo, which gives more T2*-weighting (for ky=0 echo).

- the 180°-pulse rephasing reduces the T2* signal, which is why the partially replaced asymmetric spin echo has been more commonly used.
- at higher fields, spin echo EPI is more promising
  - signal to noise is higher so we can take spin echo hit
  - contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording.
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less
  gradient power required than w/trapezoids (less eddy currents)

- earlier EPI hardware like this: sinusoidal gradient waveform
  from resonant circuit w/non-uniform sampling to get constant \( \Delta k_x \)

- sinusoids in both \( G_x \) and \( G_y \) give spiral \( k \)-space trajectory

- constant angular velocity goes too fast at large \( k_x \), \( k_y \)
- constant linear velocity better but impractical near \( k_x = 0, k_y = 0 \)
- compromise: start constant angular, end constant linear

**Constant angular velocity**

\[
\begin{align*}
  w(t) &= w_0 t \\
  k(t) &= A t e^{\imath w_0 t} \\
  G(t) &= \frac{1}{\imath} \frac{d}{dt} k(t) \\
        &= A \imath e^{\imath w_0 t} + iA w_0 e^{\imath w_0 t} \\
  G_x(t) &= A \cos w_0 t - A t w_0 \sin w_0 t \\
  G_y(t) &= A \sin w_0 t + A t w_0 \cos w_0 t
\end{align*}
\]

**Constant linear velocity**

\[
\begin{align*}
  w(t) &= w_0 T_e \\
  k(t) &= A T_e e^{\imath w_0 T_e} \\
  G(t) &= \frac{1}{\imath} \frac{d}{dt} k(t) \\
        &= \frac{A}{T_e} e^{\imath w_0 T_e} + \frac{A}{2} w_0 e^{\imath w_0 T_e} \\
  G_x(t) &= \frac{A}{T_e} \cos w_0 T_e + \frac{A}{2} w_0 \cos w_0 T_e \\
  G_y(t) &= \frac{A}{T_e} \sin w_0 T_e + \frac{A}{2} w_0 \sin w_0 T_e
\end{align*}
\]
SPIRAL 3D IR FSE  
(from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)

- Possible to present sign
- High, uniform contrast, but lots of waiting (T1), high BW

RF

180° (prep1)  T1 = 700 msec

G2

180°

Gy

180° x 16

Gx

180° (prep2)

Signal

Loop order

3D k-space
("stack of spirals")

Spiral interleave
k2 echoes

Echos \(\rightarrow\) (after one 90°)
COIL FALL-OFF / UNDERSAMPLE / GRAPPA / SENSE

- Coil fall-off intuitively contains info about position if same brain location imaged by different coils w/ diff. fall-offs

  ➔ but what does this look like in k-space?

- Slow variation in RF field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space

  ➔ to see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space — at all spatial frequencies!!

- Simple example w/ "brain" consisting of one spatial freq.

<table>
<thead>
<tr>
<th>Image domain</th>
<th>Spatial freq. domain</th>
<th>k-space</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Spatial freq." /></td>
<td><img src="image3.png" alt="K-space" /></td>
</tr>
</tbody>
</table>

  - FT of k-space data "smeared" in spatial freq. space is sharp image w/ fall-off (not blurred img.)

  - "Smeared" means normally orthogonal spatial freq's leak to adj. freqs.

- GRAPPA — construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center

- SENSE — general linear inverse approach

  - N.B.: neither would work unless normally orthogonal spatial freqs. blurred!
**Phase Errors & Echo-Centering Errors**

- Anything that causes a deviation of the $B_2$ field strength from the expected value ($B_0, z + G_{x,z} x + G_{y,z} y + G_{z,z} z$) changes precession frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

**Fourier Shift Theorem**

Phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x - x_0) = \sqrt{\frac{1}{k_x}} S(k_x) e^{i 2\pi k_x x_0}$$

- Correct with shimming and $B_0$-mapping/phase unwrapping before reconstruction.

**Echo Centering Error**

- If realignment of all spins ($k_x = k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted.

- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction => magnitude image unchanged.

**Fourier Freq. Shift Theorem**

Freq. shift in freq. domain causes phase shift in spatial.

$$I(x) = \sqrt{\frac{1}{k_x}} S(k_x - k_{x_0}) e^{i 2\pi k_x x_0}$$

- Offset in spatial freq. space.
FAST SCAN ARTIFACTS

brain-induced field defects lead to phase errors

**EPI**

- $G_x$ readout gradient strong → field defects smaller percentage
  less deformation of $k_x$ (vertical stripe components)

- Gy "blips" are weak and total Gy readout time
  much longer (5 times) than standard readout (50 ms vs. 10 ms)

- an extra gradient in the $x$-direction, for example, unmaps
  phase as a function of $x$-position

- but $G_x$ big, so effect on freq.-encode direction is much
  less than on phase-encode direction, where error accumulates

  ![k-space spin-stripe displ.]

  - for a given $x$-position, the strength of the spurious gradient
    is constant, so the accumulation of phase error results
    in a shift in the $y$-direction (k-space spin-stripe displ.)

  - the phase error causes a shift in the $y$-direction
    proportional to $x$-gradient strength (= shear) but no blurring

    (N.B. Shift varies w/ $x$-position) $y \rightarrow y$

**Spiral**

- with center-out spirals phase errors accumulate
  in a radial direction

- thus, higher spatial frequencies have more error (= more shearing)

- for spurious $x$-direction gradient as above, there is
  a radial blurring, rather than a vertical shift
  because higher frequency phase stripes misaligned
  relative to low spatial freq.

  ![k-space spin-stripe displ.]

  $y \rightarrow y$

- for defects with more complex contours in the $y$-direction
  (than linear, as above) the vertical shifts (in EPI)
  will vary with $y$-position, and may result in signals from different
  $y$-positions being reconstructed on top of each other
**Image-Space View of Localized Bφ Defect, Effect on Recon**

- Localized Bφ defects often arise from air pockets embedded in tissue
  - Air in middle/outer ear → indentation in inferior temporal lobe
  - Air under olfactory epithelium → orbitofrontal cix, ant, thal. compression

Collect one data (k-space) point

- Localized Bφ defect
- Complex multiply
  - = Correlate Sin(φ)es with brain

Brain structure sampled with distorted stripes
- One complex number

Reconstruction from distorted data points

- Same defect makes leftward dent in vertical phase stripes

Spatial information is lost when continuous changes in phase are flattened by Bφ defect

- Shifts can pile multiple pixels on top of each other into one bright pixel

Local estimates of ΔBφ needed to correct images

1) Fieldmap method: multiple TE's to est. local ΔBφ from φ/TE slope
2) Point-spread-function: extra phase encode to estimate P.S.F. (should be S-function)
   deconvolve distorted image in phase-encode direction
LOCALIZED BØ DEFECT, EFFECT ON RECON

- when local BØ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space.

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- however, with w/EPI, static BØ defect causes more and more local displacement of image phase stripes for each additional ky line

  - that is, later lines have greater spat. freq. offset
  - effectively stretches k-space in ky direction
  - same num samples to higher spatial freq. shrinks FOV (squishes voxels - see FOV page)

- when image is reconstructed, region with local BØ defect shifted oppositely.

- Thus, local shift effect due to combination of 3 things:

  1) static local ΔBØ defect

  2) successive increases in phase error for successive spat. freq. measurements during long EPI readout

  3) small size of ky phase encode blips relative to BØ defect (much less of this effect in freq. encode direction)

- respiration (which affect BØ) in 3D FLASH might cause similar effect within k2 partition (if successive spat. freqs.)

N.B. BØ affects image phase of all spatial frequencies. If we add, e.g., 90° phase, this means higher freq., image stripes move less since each cycle covers less space: ΔBØ
Gradient Non-linearities

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impress a linear variation onto the $z$-component of the $B$ field — $B_z$ — in the $x$, $y$, and $z$-directions.

- In practice, gradient coils are non-linear (e.g., printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.

  - A non-linear slice-select gradient will excite a curved slice.

  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently:
  - For 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!).

- This can result in errors approaching $1 \text{ cm}$ in functional overlays.

- Different coil designs have different directions of distortion (1).

- The assumption of non-Maxwellian gradients results in additional phase errors.

  - These can also be corrected since the $B_x$ and $B_y$ components are known.

  - That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction.
SHIMMING AND B0-MAPPING

- Passive iron shims inserted to flatten B0 field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the B0 field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc) (= several hundred ppm)

\[
\text{Linear shim coils impose gradients in } x, y, \text{ and } z
\]
\[
\text{Higher order shims impose higher order spherical harmonic field components (e.g. } z^2)\]

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the B0 field
- Local resonance offsets caused by B0 defects estimated from images
  \( \Rightarrow \) e.g., sample phase at multiple echo times
- Fit defective field using combination of fields generated by shim coils, then add these corrections to baseline shim currents
  \( \Rightarrow \) this only corrects spatially gradual field defects
  \( \Rightarrow \) local defects due to air in sinuses much higher order than shims

- After shimming, field map measured again
- Image voxel displacements calculated from resonance offset map are used to un warp the reconstructed magnitude image
- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)
**Navigator Echoes**

- **1D navigator**: excite beam of spins, detect movement from k-space shift of echo (e.g., from diaphragm)

  ![Diagram of 1D navigator](image)

  - RF pulse
  - $G_x$, $G_y$, $G_z$ gradients
  - Spin echo from 1D beam
  - Signal from head (just 90°)

- **3D navigator**: collect 3D sphere in k-space

  - Rotation of object $\rightarrow$ rotation of k-space amplitude pattern
  - Translation of object $\rightarrow$ phase shift of k-space phase (Fourier shift)
  - Sample at sufficient radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - Do N,S hemispheres separately (less T2*, cancel EPI-like error accumulation)

  \[
  \begin{align*}
  x(n) &= \sin\left(\frac{\pi}{N} \sin^{-1} \frac{2n - N - 1}{N}\right) \\
  y(n) &= \cos\left(\frac{\pi}{N} \sin^{-1} \frac{2n - N - 1}{N}\right) \\
  z(n) &= \frac{2n - N - 1}{N}
  \end{align*}
  \]

  (skip poles — slow rate too high)

Walsh et al. (2002) MRM 7

$\rightarrow$ equator $\rightarrow$ up, equator $\rightarrow$ down

- Can be used for prospective motion correction (rotate, translate w/gradients)
- Better estimate, because of speed, than Full TR & EPI images (27 ms vs. 2-4 sec)
- May need to smooth rot, trans estimates across time (e.g., Kalman filter)
RF FIELD INHOMOGENEITIES

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way. Variations can be used (e.g. SMASH, SENSE) or corrected.

- transmit coil inhomogeneities affect the flip angle in a spatially varying way. Potentially worse (why local transmit is rare) usu. fixed by using a large transmit coil (e.g. body coil).

- RF penetration at higher fields (higher RF frequencies)
  is less uniform:
  1) decreased RF wavelength (closer to size of head) at higher freq.
  2) increased permittivity ($\varepsilon$) and conductivity ($\sigma$) at higher field.

- the advantage of the falloff in signal seen with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain).

- different sensitivity functions from different coils can be used to scan less lines in k-space.

  **SMASH** — use weighted sensitivity profiles to form sinusoidal harmonics which are then used to fill in missing k-space lines.

  **SENSE** — subsample k-space (like SMASH) then unfold aliased images in image domain after estimating sensitivity profile.

  **SPACE RIP** — simple linear inverse after 1D FT (more general).
**Diffusion - Weighted Imaging**

- **Simple Diffusion Weighting**
  - RF \( \rightarrow \) \( 90^\circ \)
  - \( G_z \) along \( z \) dir.
  - \( G_y \) - readout
  - \( G_x \)

  - Spins acquire phase during first \( 8T \)
  - If spins diffuse (= move) along gradient by time \( T \), signal is lost because negative \( 8T \) doesn't re-phase
  - Attenuation: \( A(0) = e^{-bD} \)
    - \( b = (G^2 8T)^2 (T = \frac{8T}{3}) \)

  - "Apparent diffusion coefficient" map → calculate \( D \) image from \( b = 0 \) \( b \) = large

  - To get large \( b \), need \( G \uparrow\) & \( 8T \uparrow \) (need big \( G \)’s)
    - Long \( 8T \) gives spurious \( T_2 \)-weighting
    - Use stimulated echoes:
  - Long \( T \) allows

**Anisotropic Diffusion (Gaussian)**

- Measure \( D \) along multiple axes
- Have to measure tensor, not scalar → even for determining one primary direction

\[
D = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}
\]

- Since \( D \) is symmetric, only need 6 measurements

- Scalar diffusion = diffusion tensor measurement direction

**Diffusion Surface (Non-Gaussian)**

- Need to measure diffusion in many directions
- E.g., icosahedral subselection (126)
- Required for even two fiber directions

**Fiber Tract Mapping**

- An ill-posed problem
- E.g., constrain both ends and center point (?!)
- Need visual areas test
PERFUSION - ARTERIAL SPIN LABEL

- basic idea: tag blood below area of interest, collect control & tagged image, assume directional input flow!

Continuous ASL (CASSL) - continuously tag a plane, gradient on, blood gets adiabatically inverted as it passes through location w/coned resonant freq.

Pulsed ASL (PASL) - e.g., EP STAR, FAIR, PICOORE, QUIPSS II

tag block of tissue below slice(s)

- small diffs between control and tag (~1%) requires accurate balancing of control & tag images

- contrast problems: transit delays biggest confounding factor
- relaxation rate diffs, venous clearance (vs. microspheres, which get stuck!)

solutions for quantitative
- insert delay so all spins arrive into flow velocity capillary
- kill end of tag to reduce spatial variation of tag

QUIPSS II - quantitative perfusion

1) pre-saturate spins in target slices
2) tag - 180° pulse below slices (to control off-resonance)
3) saturate tagged block to end tag (TI)
   both tag and control
   can use train of thin slices
   pulses at top of tag band
4) EPI of spiral images of target slices (TI)
   image most distal slice last to cancel delays
   fast between slice so imaging excitation don't get interpreted as flow

\[ \Delta M \approx \text{flow} \times \left[ 2M_0 \cdot TI, e^{-\frac{TI_2}{T1A}} \right] \]

con extract flow and BOLD adjacent substrates minimize movement artifact

1) alternate tag and control, GRE TE = 30 ms
   control-tag -> flow
   control+tag -> BOLD (tag-control-tag-control...)
2) dual echo spiral
   k=0 early -> hi S/N flow
   TE = 30 ms -> BOLD
SPECTROSCOPY + IMAGE

- chemical shift: small displacement resonant freq due to shielding of target nucleus (e.g., $^1H$) by surrounding electron orbitals.

- e.g., acetic acid: Oxygen attracts electron so less shielding of target nucleus.

- how we get chemical shift spectrum:

  - N.B.: opposite "direction" of FTs!

<table>
<thead>
<tr>
<th>signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>time-domain samples of the mix</td>
<td>FT shift spectrum</td>
</tr>
<tr>
<td>of shifted-freq offsets</td>
<td>(shift)</td>
</tr>
<tr>
<td>spatial FT</td>
<td>spatial object (like time domain signal)</td>
</tr>
</tbody>
</table>

Pulse Sequence

- since we are already using phase (freq) encoding for space, we need an "extra dimension" w/ all gradients OFF!

- use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal

$\rightarrow$ and FT-it like chemists do!
**PHASE-ENCODED STIMULUS & ANALYSIS**

**Periodic stimuli (phase-encoded)** - e.g., 8 cycles at 64 sec/cycle

**Calculate significance**
- Ratio between amplitude at stimulus frequency (= signal) and average of amplitudes at other frequencies (= noise)
- Ignore harmonics, low freq (= movement)

**Smooth**
- Vector average of complex significance (A, φ) with that at nearest neighbor surface points

**Display**
- Plot phase using hue and saturation to indicate significance

**Delay correction**
- Record responses to opposite directions of stimulus (ccw/cw, in/out, up/down)
- Vector average after reversing angle of one 6
- Penalizes inconsistent more than just avg of angles

**Typically 0.5-5% amplitude**

**Strongly periodically activated single voxel**
**Time course**

**Remove constant (avg) and linear trend**

**FFT, convert to A, φ**

**Freq = total TRs/2**

**Reversed CCW**
**Vector average**
**CW significance**

**CCW significance (complex)**
**CONVOLUTION**

\[ f(x) = g(x) \ast h(x) = \int_{-\infty}^{+\infty} g(z) \cdot h(x-z) \, dz \]

- **Definition of Convolution**

**Why we reverse**

**Impulse Response Function (HRF)**

**Impulses (ERP design)**

- How to calculate one term
- Sum across all \( z \) to get the value of the convolution at \( x \)
- Move kernel to calculate next \( x \)

N.B., cross-corr same as convolution except no reversal
\( h(x+z) \) instead of \( h(x-z) \)

How to calculate convolution for this time point (only 3 terms in integral — all other zero)
**GENERAL LINEAR MODEL**

\[
\vec{y} = X\vec{h} + S\vec{b} + \vec{n}
\]

- Goal is to solve for the hemodynamic response functions, \( \vec{h} \)

\[
\begin{bmatrix}
\text{data} \\
\text{N}
\end{bmatrix} = \begin{bmatrix}
\text{themo} \\
\text{h}
\end{bmatrix} + \begin{bmatrix}
themo \\
\text{S}
\end{bmatrix} + \begin{bmatrix}
\text{exp} \\
\text{n}
\end{bmatrix}
\]

- \( \text{data} \) = design * HDR + drift · weights + noise

\[
\begin{bmatrix}
t_{\text{exp}} \\
\text{y}
\end{bmatrix}_{N} = \begin{bmatrix}
\text{themo} \\
\text{h}
\end{bmatrix} + \begin{bmatrix}
themo \\
\text{S}
\end{bmatrix} + \begin{bmatrix}
\text{exp} \\
\text{n}
\end{bmatrix}
\]

\( \text{y} \) = \( X \) \( \vec{h} \) + \( S \) \( \vec{b} \) + \( \vec{n} \)

\[
\text{cond 1 occurs} \\
\text{cond 2 occurs} \\
\text{cond 1 re-occurs}
\]

\[
\begin{bmatrix}
h_1 \\
h_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
h_1 \\
h_2
\end{bmatrix}
\]

\[
\text{maximum likelihood estimate}
\]

1) Assume white noise, solve for \( \hat{\vec{h}} \)

2) \( \hat{\vec{h}} = (X^T P_s^+ X)^{-1} X^T P_s^+ y \) where \( P_s^+ = I - S(S^T S)^{-1} S^T \)

3) Significance (how to construct F-ratio) 

\[
F = \frac{N-K-L}{K} \left[ \frac{y^T (P_{Xs} - P_s) y}{y^T (I - P_{Xs}) y} \right]
\]

- see diagram next page for geometric interp
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- with no nuisance functions ($S$), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

\[ \hat{y} = Xh + \hat{e} \]

Projection matrix, $P_x$, operates on $\hat{y}$ to give projection of data into experiment space, $X$

- when nuisance functions, $S$, are considered, problem: $S$ may not be orthogonal to $X$

\[ P_y \]

for example: linear trend not orthogonal to std. block design

[remember: "orthogonal" means $\text{dot prod.} = 0$

$\text{corr} = 0$

$\Rightarrow \text{sum. out. not 0}$

Geometric Picture

(Liu et al. 2001, Neuroimage)

$Xs$: space of data modeled by all reference and nuisance

$E_y$: data orthogonal to nuisance

$P_y$: data explained by reference

$P_{xy}$: orthogonal projection onto nuisance

error ($e$) not explained by reference and nuisance

$[(I - P_{ys})y]$

how much more of data you can explain by adding reference functions (F numerator)

$[(P_{xs} - P_s)y]$

same as projection onto reference only in special case where $S \perp X$
1) MNI auto-Talairach → generates 4×4 matrix
   - make average brain target (blurry)
   - blur target (further), blur single brain (a bit), gradient descent on xcorr
   - repeat w/ less blurring of avg target and current brain
   - problems: variable neck cut-off
     → but much better than standard! fit to bounding box
   only 2 points near center of brain!

2) Intensity Normalization (output: "T1")
   - histogram of pixel values in 10 mm thick HR slices
   - smooth histogram
   - peak find to get initial estimate of white matter
   - discard outlier peaks across slices
   - fit splines to peaks across slices
     → interprets as scaling factor 1 to HR
   - scale each pixel so WM peak is 110
   - refine estimate to interpolate in 3D
     - find points in 5×5×5 within 10% of WM, get new scale for them
     - build Voronoi to interpolate scales, set above
     - soap-bubble, smooth Voronoi boundaries (3 iterations)
     - re-scale each voxel
     5-10 times

3) Skull Stripping (output: "brain")
   - "shrink-wrap" algorithm
   - start with ellipsoidal template → sub-tessellated icosahedron
   - minimize brain penetration and curvature
     - curvature: spring force
       (from center-to-neighbor vect sum)
   - brain penetration
     - apply force along surface normal that
       prevents surface from entering gray matter
     - decompose into 1 and tangential (local normal from summed
       normed cross products)
SEGMENTATION & SURFACE RECON
Spring force in detail

- Implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something")
- More formally, we would define cost function, then take its derivative (gradient) to minimize it.

Shrinkwrap update eq. (skull strip, original Dale & Sereno surface refinement)

\[ \mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t) \]

**Rule for each vertex, \( \mathbf{r}_{\text{center}} \)**

\[ \mathbf{F}_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (\mathbf{I} - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \]

Identify 3x3 stronger than normal (0.5)

Identify 3x3 vector to neighbor vertex

Expand by distribution to neighbor vertex minus projection of neighbor onto normal = tangential!

\[ + \lambda_{\text{normal}} \sum_{\text{neigh}} (\mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) - \frac{1}{\#\text{vertices}} \sum_{\text{neigh}} \sum_{\text{v}} (\mathbf{n}_v \mathbf{n}_v^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_v) \]

Project vector onto normal in the direction of the normal (\( \mathbf{n} \)) is squared (as above) so we get a vector out (not a scalar)

Average normal component

\[ \mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \frac{30}{d} \max \left[ 0, \tanh \left[ I \left( \mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}} \right) - I_{\text{threshold}} \right] \right] \]

Multiply all terms \( \mathbf{F}_{\text{MRI}} = 0 \) if any are zero

Max force saturates at 1.0 don't allow max product = 1.0

Sample points into brain along the direction of normal

Surface (moving outward)

Snapshot of surface and "core sample" from one vertex

GM

WM

Ideal skull strip

Outside (dark)

Skin (light)

Skull (dark-light; dark)
SEGMENTATION & SURFACE RECON

4) Non-isotropic filtering (output: "thin") — "floss" and "sponge"
- preliminary hard thresholding:
  - find ambiguous/boundary voxels
    \[ \Rightarrow \text{20\% or more of 26 immediate neighbors different} \]
    \[ \Rightarrow \text{to avoid expensive calc below...} \]
  - find plane of least variance
    \[ \text{for each direction (from icosahedral super-tessellation)} \]
    \[ \text{consider 5x5x5 volume around 1 voxel} \]
    \[ \text{find plane of least variance in this hemisphere} \]
    \[ \text{median filter w/ hysteresis} \]
    \[ \Rightarrow \text{if 60\% of within-slab differ, reverse classification} \]
    \[ \Rightarrow \text{"flosses" sulci without blurring} \]

5) Find cutting planes
- midbrain
- callosum, to separate hemispheres (SAH)
- midbrain, to avoid fill into cerebellum (THOR)
- Talairach to start;
  fill WM in SAH or THOR till min area

6) Region-growing to define connected parts (output: "filled")
- inside-out, outside-in, inside-out
- up/down cycles within each plane
- plane-by-plane
  \[ \Rightarrow \text{"wormhole filter" (3x3x3 = center + 26)} \]
  \[ \Rightarrow \text{fill (unfilled) voxel if 66\% neighbors differ} \]
  \[ \Rightarrow \text{eliminates structures within 1-D structure} \]
7) **Surface Tessellation** (output: rh.orig, lh.orig)

- **variable num neighbors possible!**
- **quads to triangles**

- find filled voxels bordering unfilled
- make ordered list of neighboring vertices
  - so cross-products oriented properly

- long list of values associated with each numbered vertex
  - e.g.: position (orig, morphed)
  - area (orig, morphed)
  - curvature (intrinsic, Gaussian)
  - "sulcushness" (summed ± movement during unfolding)
  - cortical thickness
  - **fMRI data**
  - EEG/MEG dipole strength

- separate fMRI data set must be aligned, sampled
  - fMRI voxels larger
  - Sample at each surface vertex
    - nearest-neighbor "soap bubble" smoothing
      - to interpolate data onto hi-res mesh
- some quantities only well-defined on surface
  - gradient of magnitude of cortical map measure (e.g., eccentricity)
- smoothing/inflation WM, pial done as derivative of energy functional

\[ J = \frac{1}{2} \sum \sum \left[ \frac{d}{dr} \cdot (r_{\text{center}} - r_{\text{neighbor}})^2 \right]^{1/2} \]

\[ J_{\text{normal}} = \frac{1}{2} \sum \sum \frac{n_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}})}{2} \]

\[ J_{\text{tangential}} = \frac{1}{2} \sum \sum \left[ t^x_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}}) \right] + \left[ t^y_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}}) \right] \]

\[ J_{\text{image}} = \frac{1}{2} \sum \left[ I_{\text{center}} - I_{\text{target}} \right]^2 \]

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-interest test

\[ \frac{\partial J}{\partial r_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{center}} - I_{\text{target}} \right] \nabla I (r_{\text{center}}) \]

\[ + \sum \lambda_{\text{normal}} \frac{n_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}})}{2} \]

\[ + \sum \left[ t^x_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}}) \right] t^x_{\text{center}} + \left[ t^y_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}}) \right] t^y_{\text{center}} \]

N.B.: eq. 9 in Dalo, Fisch & Sarem different — and incorrect!
SULCUS-BASED CROSS-SUB. ALIGN

- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"
- Add term to energy function: "sulcus-ness" error: \( (S_{\text{ent}} - S_{\text{targ}})^2 \)
- Bootstrap morph to one brain make avg target remorph to avg target

Smooth wm inflated sphere registered sphere

Sub_1

Sub_2

Sub_n

- Each sub's native surf has diff # vertices
- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)

- Average surface made from folded/inflated avg coords
  Folded: Loses area from sulcal crinkles (\( f_{\text{average}} "\text{inflated}" \))
  Inflated: Retains orig area, correct sulc gyrus ratio ("inflated-avg")

- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

\( \rightarrow \) N.B.: Morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)